

On convexification/optimization of functionals including an l^2 -misfit term

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Abstract

We provide theory for computing the (lower semicontinuous) convex envelope of functionals of the type

$$f(x) + \frac{1}{2}\|x - d\|^2, \quad (1)$$

by introducing a new transform, and discuss applications to various non-convex optimization problems. The latter term is a data fit term whereas f provides structural constraints on x . By minimizing (1), possibly with additional constraints, we thus find a tradeoff between matching the measured data and enforcing a particular structure on x , such as sparsity or low rank. For these particular cases, the theory provides alternatives to convex relaxation techniques such as ℓ^1 -minimization (for vectors) and nuclear norm-minimization (for matrices). In cases when the convex envelope is not explicitly computable, we provide theory for how minimizers of (explicitly computable) approximations of the convex envelope relate to minimizers of the original functional. In particular, we give explicit conditions on when the two coincide.

Keywords: Fenchel conjugate, convex envelope, non-convex/non-smooth optimization

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1. Introduction

The purpose of this article is to convexify, or partially convexify, functionals of the type

$$\|x\|_0 + \frac{1}{2}\|Ax - d\|_2^2 \quad (2)$$

where $x \in \mathbb{R}^n$, and

$$\text{rank}(X) + \frac{1}{2}\|X - D\|_F^2 \quad (3)$$

where $X \in \mathbb{M}_{m,n}$ (the space of $m \times n$ -matrices with the Frobenius norm). In other words, we are interested in computing the l.s.c. convex envelope or at least an approximation thereof. We will also consider weighted norms and penalty terms like

$$f(X) = \begin{cases} 0 & \text{rank}(X) \leq K, \\ \infty & \text{else.} \end{cases} \quad (4)$$

in order to treat problems where a matrix of a fixed rank is sought. Such functionals appear in a multitude of optimization problems, where the goal is to find a point x such that the

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functional attains its minimum, possibly with additional constraints on x . We refer to the overview article [26] which includes a long list of applications. The problem of minimizing (2) and (3) differ significantly in that (3) has a closed form solution whereas solving (2) is NP-hard. However, minimization of (3) over a subspace or in combination with additional priors, is also a hard well-known problem with many applications, and knowing the l.s.c. convex envelope can help to find approximate solutions, as we advocate in this paper. We refer to [18, 20] and the references therein for examples of applications.

Since the functional (4), as well as $\|\cdot\|_0$ and $\text{rank}(\cdot)$, are non-convex, it is tempting to replace them by their convex envelopes. However, in all three cases the convex envelope equals 0. To obtain problems that are efficiently solvable, it is therefore popular to replace e.g. $\|\cdot\|_0$ with the ℓ^1 -norm or $\text{rank}(\cdot)$ by the nuclear norm, a strategy which is sometimes called convex relaxation, thus obtaining a convex problem reminiscent of the original problem. Such methods have a long history, but has received new attention in recent times due to the realization that the original problem and the convex relaxation under certain assumptions have the same solution, as pioneered in the work concerning compressed sensing [9]. The argument behind the choice of convex replacement is often that the functionals in question are the convex envelopes of the original ones when restricted to the unit ball, see e.g. [20].

Despite the success of these methods, there is a huge difference between the functional $\|x\|_0$ and $\|x\|_1$ for large values of x , which usually leads to a bias in the solution of the convex relaxation, which is a well known issue. We refer to e.g. [18, 25] for a deeper discussion and further references concerning these problems. To remedy this, there has recently been two independent attempts at finding local convexifications closer to the original functional, namely [18] for minimizing (3) (in combination with additional restrictions) and [25] for minimizing (2) as is. In this paper we find a unifying framework and significantly extend the existing theory.

Figure 1 highlights these issues; let χ_A be the characteristic functional of a set A , i.e. the function equalling 1 on A and zero elsewhere. In red we see the functional $\chi_{\mathbb{R}}(x) + \frac{1}{2}|x - 1|$ where $\mathbb{R} = \mathbb{R} \setminus \{0\}$ (which is a particular case of both (2) and (3) in dimension 1), in blue its convex envelope and in pink the convex relaxation $|x| + \frac{1}{2}|x - 1|^2$. Clearly the global minimum of the red and blue coincide, but the global minimum of the convex relaxation is different. We will return to this picture in Section 4.1.

We now outline the main contributions of this paper in greater detail. Consider any functional of the form

$$f(x) + \frac{1}{2}\|x - d\|_{\mathcal{V}}^2 \quad (5)$$

where \mathcal{V} is an arbitrary separable Hilbert space and f any functional on \mathcal{V} that is bounded below. We introduce a new transform – the \mathcal{S} -transform – denoted $\mathcal{S}(f)$ or f° , and show that

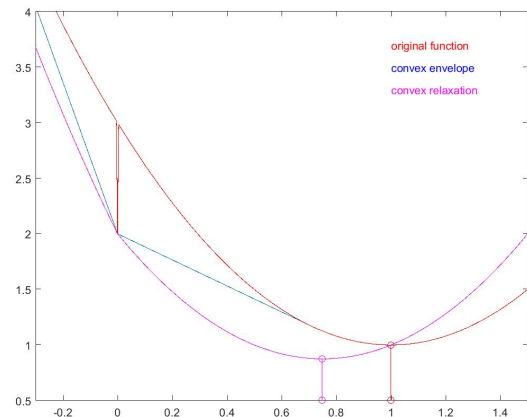


Figure 1: Illustration of a non-convex, non-continuous functional together with its convex envelope and a “traditional” convex relaxation.

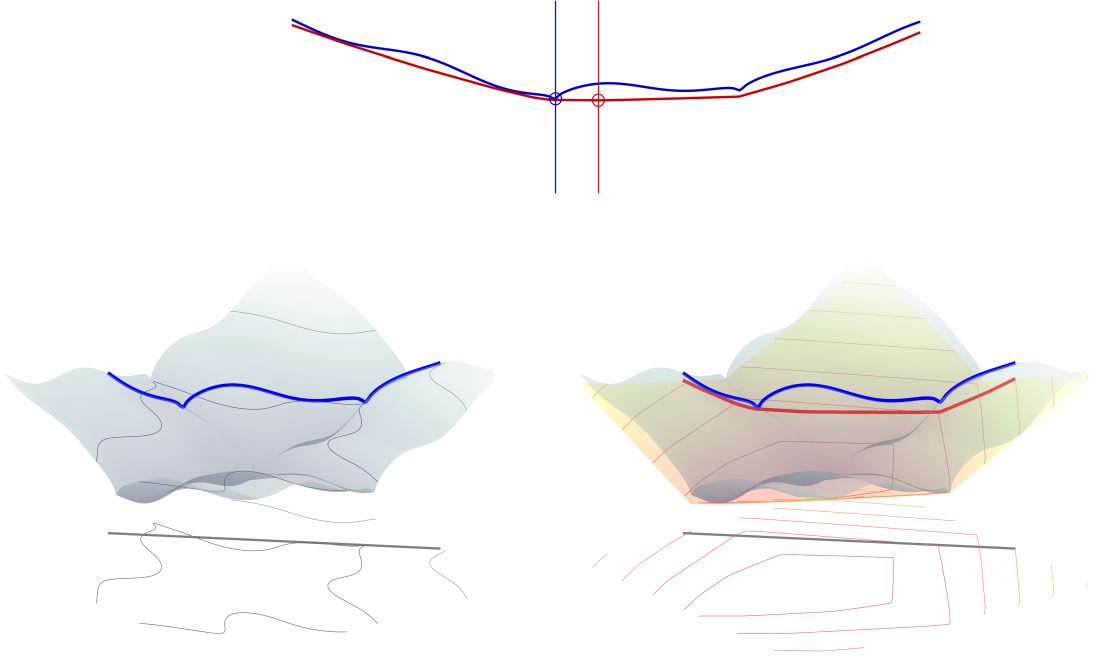


Figure 2: Illustration of a non-convex optimization problem with linear constraints. The bottom left panel shows a non-convex functional along with its level sets. The gray line represents the subspace we are interested in, and the blue curve the values of the functional restricted to the subspace. The bottom right panel shows the same setup, but here the convex envelope is shown as well in orange/yellow. The values of the convex envelope over the subspace is shown in the red curve. The top figure shows a one-dimensional plot of the values of the original functional (blue) and the convex envelope (red) evaluated on the subspace. The respective minima are shown by circles and highlighted by the vertical lines. Note that they are located close to each other, but that they are not identical despite the fact that the global minimum for the original functional and its convex envelope coincide.

the (l.s.c) convex envelope of the functional in (5) is

$$f^{\circ\circ}(x) + \frac{1}{2}\|x - d\|_{\mathcal{V}}^2, \quad (6)$$

where $f^{\circ\circ} = \mathcal{S}(\mathcal{S}(f))$ (Section 2.1). Note in particular that the shape of the convex envelope is completely independent of d . The functionals f° and $f^{\circ\circ}$ are closely related to the Moreau-envelope and the Lasry-Lions approximants, which we elaborate more on in Section 2.1. In Section 2.2 we provide numerous examples of $f^{\circ\circ}$ for various functionals acting on matrices as well as vectors. We also provide a number of general results to simplify the computation of $f^{\circ\circ}$. Finer properties of the \mathcal{S} -transform are proven in Section 2.3, which concludes the first part of the paper, titled “general theory”.

The remainder of the paper is divided in two parts corresponding to the prototype functionals (2) and (3), which are very different in nature. To further explain why, we remark that f° can be computed explicitly only if the global minimum of the original functional (5) can be found explicitly, as in the case of (3). Therefore, as opposed to the situation in (2), the problem only becomes difficult in combination with additional restrictions. Suppose e.g. that we want to minimize (5) over some subspace $\mathcal{M} \subset \mathcal{V}$ or say that we wish to minimize

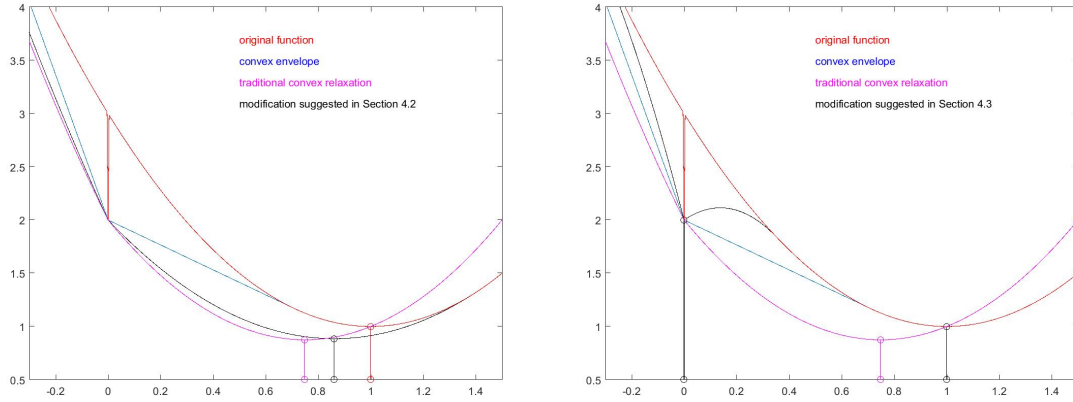


Figure 3: The same setup as in Figure 1, but with an additional functional in black illustrating (8) in case A is expansive (left) and in case A is contractive (right). See Section 4.1 for a more detailed description.

$f(x) + \|x - d\|^2 + c(x)$ where c is a convex functional related to any additional prior information, (see Section 4 in [18] for concrete examples). In both cases we end up with minimization problems with no closed form solution. Replacing f with $f^{\circ\circ}$ then gives us a convex problem, similar to the original one, which can be addressed with standard convex approximation schemes like the projected subgradient method, dual ascent, ADMM or forward-backward splitting. It is often the case that the minimum of the “convexified” problem coincides with the minimum of the non-convex problem, which is easily verified by simply checking if $f(x) = f^{\circ\circ}(x)$ holds at the point of convergence. It is important however to realize that this is not always the case, as Figure 2 demonstrates. However, Figure 7 in Section 3.1 shows the same functional with a different subspace in which the two minima does coincide. We elaborate further on this in Section 3.1. More information on these issues as well as the rationale behind replacing f by $f^{\circ\circ}$ is also found in [18] (specific to certain rank type functionals acting on matrices) and [3] (studying convex envelopes in greater generality and dual ascent). It is not the aim of the present paper to provide recommendations for which algorithm to use to solve a specific application, and the best candidate will certainly depend on the particular situation. Nevertheless, several of the algorithms mentioned above requires the ability to compute the so called proximal operator, and we provide theory for this in Section 3.3, which concludes Part II of the paper, titled “applications when explicit formulas are available”.

Part III of the paper is devoted to the problem of minimizing

$$f(x) + \frac{1}{2}\|Ax - d\|_{\mathcal{V}}^2 \quad (7)$$

where A is any linear transformation. We assume that \mathcal{V} is such that $f^{\circ\circ}$ is computable, but due to the linear transformation A , the functional

$$f^{\circ\circ}(x) + \frac{1}{2}\|Ax - d\|_{\mathcal{V}}^2 \quad (8)$$

will not equal the convex envelope of (7), which we assume is untractable, as in the case of (2). The expression (8) is illustrated (in one dimension and for values of $A > 1$ (left) and $A < 1$ (right)) in Figure 3, taken from Section 4.1 where further examples are given. Generalizing the left figure, we assume in Section 4.2 that A is expansive, i.e. that $\|Ax\| \geq \|x\|$ for all x .

We prove that the functional (8) is a convex functional below (7), and hence minimization of (8) will produce a minimizer which, although not necessarily equal to the minimizer of the original problem, likely is closer than that obtained by other convex relaxation methods (if such at all are available). Moreover, the minimizer of the original and modified problem do coincide whenever $f(x) = f^{\circ\circ}(x)$, which often is easily checked in practice. An example of when this happens, similar to Figure 3, is shown in Figure 8 in Section 4.2.

For the problem (2), A is usually a matrix with a large kernel, and hence it can not be expansive. In Section 4.3 we consider the case when A is a strict contractions, i.e. $\|A\| < 1$, generalizing the situation in the right picture of Figure 3. We can then show that (8) is a continuous (but not everywhere convex) functional with the following desirable properties; *i*) (8) lies between (7) and its l.s.c. convex envelope, *ii*) any local minimizer of (8) is a local minimizer of (7), *iii*) the global minimizers of (8) and (7) coincide (see Proposition 4.5 and Theorem 4.6). We remark that, despite not being convex, critical points of (8) can be found using e.g. the forward-backward splitting method [4, 7]. The situation in Section 4.3 is thus drastically different from the previous scenarios; whether a global minimizer of the original problem is found depends only on the starting point for the algorithm seeking local minimizer.

This latter part of the paper is inspired by [25], which considers problem (2), and also contains a list of recent algorithms for finding local minima of functionals of the type considered above. We briefly revisit the problem (2) in the final Section 4.4.

The paper also contains two appendices of some independent interest. Appendix I revisits an extension of Milman's theorem, giving structural properties of l.s.c. convex envelopes, due to A. Brøndsted in a short notice from 1966 [8], which seems to have remained unnoticed by the community. Appendix II extends the famous von Neumann's trace inequality to operators on infinite dimensional spaces, a result which rather surprisingly has not been published to our best knowledge.

Notation

Set $\mathbb{R} = \mathbb{R} \setminus \{0\}$. The set of $m \times n$ complex matrices, equipped with the Frobenius norm, is denoted $\mathbb{M}_{m,n}$. Throughout the paper, \mathcal{V} and sometimes \mathcal{W} denote separable Hilbert spaces (possibly finite dimensional). Let $\mathcal{B}_2(\mathcal{V}, \mathcal{W})$ denote all Hilbert-Schmidt operators with the Hilbert-Schmidt norm. We remark that in case $\mathcal{V} = \mathbb{C}^n$ and $\mathcal{W} = \mathbb{C}^m$ with the canonical norms, then $\mathcal{B}_2(\mathcal{V}, \mathcal{W})$ is readily identified with $\mathbb{M}_{m,n}$ with the Frobenius norm. The singular value decomposition (SVD) of a given $A \in \mathbb{M}_{m,n}$ is denoted $A = U\Sigma V^*$, where we choose $V \in \mathbb{M}_{n,n}$, $\Sigma \in \mathbb{M}_{n,n}$ and $U \in \mathbb{M}_{m,n}$. The vector of singular values (i.e. the elements on the diagonal of Σ) is then denoted by σ . Note that we thus define the singular values such that the amount of singular values equals the dimension of \mathcal{V}_1 . More generally, given any operator A acting on infinite dimensional spaces, we can pick singular vectors $(u_j)_{j=1}^\infty$ and $(v_j)_{j=1}^\infty$ such that

$$A = \sum_{j=1}^{\infty} \sigma_j(A) u_j \otimes v_j \quad (9)$$

where $\sigma_j(A)$ are the singular values and $u_j \otimes v_j(x) = u_j \langle x, v_j \rangle$. Moreover $(u_j)_{j=1}^\infty$ can be taken to be an orthonormal sequence in \mathcal{W} and $(v_j)_{j=1}^\infty$ to be an orthonormal basis in \mathcal{V} (see e.g. Theorem 1.4 [23]). We follow the matrix theory custom of numbering the singular vectors starting at 1, as opposed to 0 which is more common in operator theory.

$\mathbb{H}(\mathcal{V})$ will denote the subspace of $\mathcal{B}_2(\mathcal{V}, \mathcal{V})$ of self-adjoint (Hermitian) operators, and $\lambda(X)$ the vector of eigenvalues of a given $X \in \mathbb{H}(\mathcal{V})$. In case \mathcal{V} has finite dimension n , so that $\mathcal{B}_2(\mathcal{V}, \mathcal{V})$ is identified with $\mathbb{M}_{n,n}$, we simply write \mathbb{H}_n .

Given $x \in \mathbb{R}^n$, $\|x\|_0$ denotes the amount of non-zero elements (by abuse of notation since this is not a norm), and $\|x\|_2$ the canonical scalar product. We abbreviate lower semi-continuous by l.s.c., and we denote by $\text{dom}(f)$ the set of points where the functional f is finite.

2. Part I; general theory.

2.1. The \mathcal{S} -transform

Let \mathcal{V} be a separable Hilbert space, such as \mathbb{C}^n with the canonical norm $\|x\|_2^2 = \sum_{j=1}^n |x_j|^2$ or $\mathbb{M}_{m,n}$, equipped with the Frobenius norm which we denote $\|X\|_F$. Given any functional $g : \mathcal{V} \rightarrow \mathbb{R} \cup \{\infty\}$ the Legendre transform, denoted $\mathcal{L}_{\mathcal{V}}$, is defined as

$$\mathcal{L}_{\mathcal{V}}(g)(y) := \sup_x \text{Re} \langle x, y \rangle - g(x). \quad (10)$$

If \mathcal{V} is understood from the context we usually write $\mathcal{L}_{\mathcal{V}}(g) = g^*$. We recall the following well known properties of Legendre transforms, see e.g. Proposition 13.11 and 13.39 in [5].

Proposition 2.1. *Let g be a $[0, \infty]$ -valued functional on a separable Hilbertspace \mathcal{V} . Then g^* is l.s.c convex and g^{**} equals the l.s.c. convex envelope of g .*

Given a functional f on \mathcal{V} we now introduce a transform $\mathcal{S}_{\mathcal{V}}$ defined as follows

$$\mathcal{S}_{\mathcal{V}}(f)(y) := \mathcal{L}_{\mathcal{V}} \left(f(\cdot) + \frac{1}{2} \|\cdot\|_{\mathcal{V}}^2 \right) (y) - \frac{1}{2} \|y\|_{\mathcal{V}}^2 = \sup_x -f(x) - \frac{1}{2} \|x - y\|_{\mathcal{V}}^2. \quad (11)$$

When \mathcal{V} is clear from the context, we will usually write f° instead of $\mathcal{S}_{\mathcal{V}}(f)$, or abbreviate $\mathcal{S}_{\mathcal{V}}$ by \mathcal{S} . We remark that $f^{\circ\circ}(x) + \frac{1}{2} \|x\|_{\mathcal{V}}^2$ is the convex envelope of $f(x) + \frac{1}{2} \|x\|_{\mathcal{V}}^2$, which is immediate by iteration of

$$\mathcal{L} \left(f(\cdot) + \frac{1}{2} \|\cdot\|_{\mathcal{V}}^2 \right) (y) = f^\circ(y) + \frac{1}{2} \|y\|_{\mathcal{V}}^2. \quad (12)$$

It is clear from the second line of (11) that f° is simply the negative of the famous Moreau-envelope. However, the double Moreau-envelope does not equal $f^{\circ\circ}$, and is not connected with convex envelopes. For $s = t = 1$ we do have

$$f^{\circ\circ}(x) = - \left(\min - \left(\min_w g(w) + \frac{t}{2} \|w - y\|_{\mathcal{V}}^2 \right) + \frac{s}{2} \|x - y\|_{\mathcal{V}}^2 \right),$$

i.e. the negative of the Moreau envelope of minus the Moreau envelope (of f) does equal $f^{\circ\circ}$. For general parameters $s > t$, the above functional is called the Lasry-Lions approximation of f , which has been extensively studied in the context of regularization of non-convex functionals, see e.g. . The connection with convex envelopes seems to have been ignored, and it is the aim of this paper to systematically study this topic and its applications. We begin by noting some simple structural properties.

Proposition 2.2. *Let f be a $[0, \infty]$ -valued l.s.c. functional on a separable Hilbert space \mathcal{V} . Then f° takes values in $(-\infty, 0]$ and is continuous, whereas $f^{\circ\circ}$ takes values in $[0, \infty]$ and is continuous in the interior of $\text{dom}(f^{\circ\circ})$.*

Proof. The statements of the interchanging signs follows easily by the last line of (11), which also shows that f° avoids $-\infty$. By Proposition 2.1 and (12) it follows that f° (and $f^{\circ\circ}$) is the difference of an l.s.c. convex functional and a quadratic term. With this in mind the continuity statements follows by standard properties of l.s.c. convex functionals (see e.g. Corollary 8.30 [5]). \square

We now focus on the convex envelope of $f(x) + \frac{1}{2}\|x - d\|_{\mathcal{V}}^2$ for some fixed $d \in \mathcal{V}$.

Theorem 2.3. *Let f be a $[0, \infty]$ -valued functional on a separable Hilbert space \mathcal{V} . Then*

$$\mathcal{L}_{\mathcal{V}} \left(f(x) + \frac{1}{2}\|x - d\|_{\mathcal{V}}^2 \right) (y) = f^\circ(y + d) + \frac{1}{2}\|y + d\|_{\mathcal{V}}^2 - \frac{1}{2}\|d\|_{\mathcal{V}}^2$$

and

$$\mathcal{L}_{\mathcal{V}} \left(f^\circ(y + d) + \frac{1}{2}\|y + d\|_{\mathcal{V}}^2 - \frac{1}{2}\|d\|_{\mathcal{V}}^2 \right) (x) = f^{\circ\circ}(x) + \frac{1}{2}\|x - d\|_{\mathcal{V}}^2.$$

In particular, $f^{\circ\circ}(x) + \frac{1}{2}\|x - d\|_{\mathcal{V}}^2$ is the l.s.c. convex envelope of $f(x) + \frac{1}{2}\|x - d\|_{\mathcal{V}}^2$ and $0 \leq f^{\circ\circ} \leq f$.

Proof. We omit explicit reference to \mathcal{V} in the notation. Then

$$\begin{aligned} \mathcal{L} \left(f(x) + \frac{1}{2}\|x - d\|^2 \right) (y) &= \sup_x \langle x, y \rangle - f(x) - \frac{1}{2}\|x - d\|^2 = \\ &= \sup_x \text{Re} \langle x, y + d \rangle - f(x) - \frac{1}{2}\|x\|^2 - \frac{1}{2}\|d\|^2 = f^\circ(y + d) + \frac{1}{2}\|y + d\|^2 - \frac{1}{2}\|d\|^2. \end{aligned}$$

Similarly

$$\begin{aligned} \mathcal{L} \left(f^\circ(y + d) + \frac{1}{2}\|y + d\|^2 - \frac{1}{2}\|d\|^2 \right) &= \sup_y \text{Re} \langle x, y \rangle - f^\circ(y + d) - \frac{1}{2}\|y + d\|^2 + \frac{1}{2}\|d\|^2 = \\ &= \sup_y \left(\text{Re} \langle x, y + d \rangle - f^\circ(y + d) - \frac{1}{2}\|y + d\|^2 \right) + \frac{1}{2}\|d\|^2 - \text{Re} \langle d, y \rangle = \\ &= f^{\circ\circ}(x) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|d\|^2 - \text{Re} \langle d, y \rangle = f^{\circ\circ}(x) + \frac{1}{2}\|x - d\|^2. \end{aligned}$$

The statement about convex envelope is a direct consequence of Proposition 2.1, by which we immediately get $f^{\circ\circ}(x) + \frac{1}{2}\|x - d\|^2 \leq f(x) + \frac{1}{2}\|x - d\|^2$. This implies the latter part of the inequality $0 \leq f^{\circ\circ} \leq f$, whereas the former has already been noticed Proposition 2.2. \square

The above theorem can also be applied to expressions of the form

$$f(x) + \frac{1}{2}\|Ax - d\|_2^2, \quad x \in \mathbb{C}^n \tag{13}$$

upon renormalizing \mathcal{V} using A , but we postpone the theory for this case to Part III, in particular Proposition 4.8. Finer properties of the \mathcal{S} -transform are discussed in Section 2.3. In the coming section we make a long list of computable \mathcal{S} -transforms as well as provide general tools to compute such.

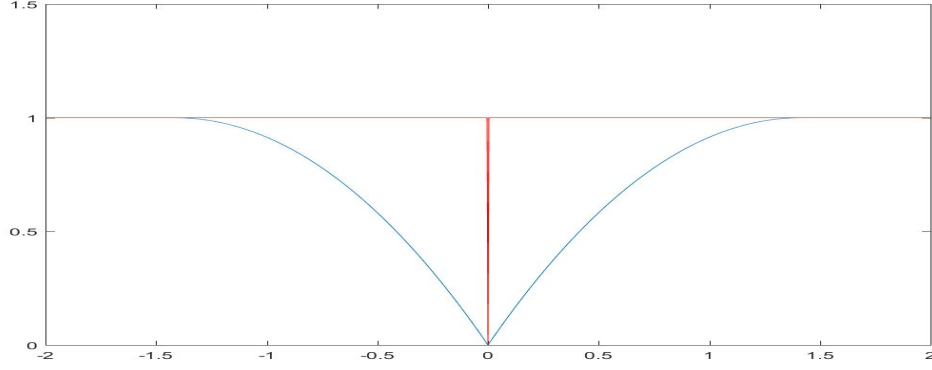


Figure 4: Illustration of $\chi_{\mathbb{R}}(x)$ (red) along with its double S -transform $(\chi_{\mathbb{R}}(\cdot))^{\circ\circ}(x)$ for $\nu = 1$ in Example 2.4.

2.2. Examples of \mathcal{S} -transforms

We begin by studying the functional $\chi_{\dot{\mathbb{R}}}(x)$, where χ_S denotes the characteristic functional of a set S and $\dot{\mathbb{R}} = \mathbb{R} \setminus \{0\}$. This seemingly innocent functional is relevant for both key problems (2) and (3), which follows by noting that

$$\|x\|_0 = \sum_{j=1}^n \chi_{\dot{\mathbb{R}}}(x_j) \quad (14)$$

and

$$\text{rank}(X) = \sum_{j=1}^n \chi_{\dot{\mathbb{R}}}(\sigma_j(X)). \quad (15)$$

Example 2.4. Let $\mathcal{V} = \mathbb{R}$ with norm $\|x\|_{\mathcal{V}}^2 = \nu^2|x|^2$ and set $f(x) = \tau^2\chi_{\dot{\mathbb{R}}}(x)$ where $\nu, \tau > 0$ are fixed parameters (see red curve in Figure 4). Then

$$f^\circ(y) = \mathcal{S}(f)(y) = \sup_x -\tau^2\chi_{\dot{\mathbb{R}}}(x) - \frac{\nu^2}{2}(x-y)^2. \quad (16)$$

Clearly, the maximum is found either at $x = 0$ or at $x = y$ which gives

$$f^\circ(y) = \sup\left\{-\frac{\nu^2 y^2}{2}, -\tau^2\right\} = -\min\left\{\frac{\nu^2 y^2}{2}, \tau^2\right\}. \quad (17)$$

To compute $f^{\circ\circ}$, we repeat the process

$$f^{\circ\circ}(x) = \sup_y -\left(-\min\left\{\frac{\nu^2 y^2}{2}, \tau^2\right\}\right) - \frac{\nu^2}{2}(x-y)^2 = \sup_y \min\left\{\frac{\nu^2 y^2}{2}, \tau^2\right\} - \frac{\nu^2}{2}(x-y)^2$$

Since $\min\left\{\frac{\nu^2 y^2}{2}, \tau^2\right\}$ is constantly equal to its supremum value τ^2 whenever $|y| \geq \sqrt{2}\tau/\nu$, it follows that the maximum is attained at $y = x$ for all x satisfying $|x| \geq \sqrt{2}\tau/\nu$, yielding $f^{\circ\circ}(x) = \tau^2$. For the same reason the maximum is attained in $[-\sqrt{2}\tau/\nu, \sqrt{2}\tau/\nu]$ whenever

$|x| < \sqrt{2}\tau/\nu$. Since the y^2 -terms cancel in this segment, the functional to be maximized is linear there, and so the maximum must be obtained at $y = \pm\sqrt{2}\tau/\nu$. It easily follows that

$$f^{\circ\circ}(x) = \tau^2 - \frac{\nu^2}{2}(|x| - \frac{\sqrt{2}\tau}{\nu})^2 \chi_{[-\frac{\sqrt{2}\tau}{\nu}, \frac{\sqrt{2}\tau}{\nu}]}(x) = \tau^2 - \left(\max\{\tau - \frac{\nu|x|}{\sqrt{2}}, 0\}\right)^2. \quad (18)$$

For the values $\nu = \tau = 1$ this is shown in blue in Figure 4, and the functional $f^{\circ\circ}(x) + \frac{1}{2}|x-1|^2$ can be seen in blue in Figure 1.

The above simple example allows us to compute the \mathcal{S} transform of the more complicated cost functionals (14) and (15), when combined with the below propositions. We refer to Ch. I.6 in [12] for the basics of direct products of separable Hilbert spaces. We write $\mathcal{S} = \mathcal{S}_{\mathcal{V}}$ if there is a need to clarify which space is used to compute the transform.

Proposition 2.5. *Let $(\mathcal{V}_j)_{j=1}^d$ where $d \in \mathbb{N} \cup \{\infty\}$ be separable Hilbert spaces and set $\mathcal{V} = \oplus_{j=1}^d \mathcal{V}_j$. Suppose that f_j are $[0, \infty]$ -valued functionals on \mathcal{V}_j and set $F(x) = \sum_{j=1}^d f_j(x_j)$ where $x = \sum_{j=1}^d x_j$ and $x_j \in \mathcal{V}_j$. Then*

$$\mathcal{S}_{\mathcal{V}}(F)(y) = \sum_{j=1}^d \mathcal{S}_{\mathcal{V}_j} f_j(y_j).$$

Proof. We have that

$$F^{\circ}(y) = \sup_x -F(x) - \|x - y\|_{\mathcal{V}}^2 = \sup_x \sum_{j=1}^d -f_j(x_j) - \|x_j - y_j\|_{\mathcal{V}_j}^2 = \sum_{j=1}^d f_j^{\circ}(y_j).$$

□

Combining this with Example 2.4 we immediately get

$$\|x\|_0 = \sum_{j=1}^n 1 - \left(\max\{1 - \frac{|x|}{\sqrt{2}}, 0\}\right)^2 \quad (19)$$

To derive a similar expression for (15), we need von-Neumann's trace inequality for operators on separable Hilbert spaces. We thus shift focus to functionals acting on the singular values of a matrix or, more generally, a Hilbert-Schmidt operator in $\mathcal{B}_2(\mathcal{V}_1, \mathcal{V}_2)$. Set $n = \dim \mathcal{V}_1$ and note that the singular values of a given $X \in \mathcal{B}_2(\mathcal{V}_1, \mathcal{V}_2)$ lie in the set \mathbb{R}^n , which we identify with $\ell^2(\mathbb{N})$ in case $n = \infty$ (see the Notation section for conventions concerning numbering of singular values).

Given two separable Hilbert spaces $\mathcal{V}_1, \mathcal{V}_2$, we let $\mathcal{B}_2(\mathcal{V}_1, \mathcal{V}_2)$ be the Hilbert space of Hilbert-Schmidt operators with the standard norm (see e.g. [23]). The inequality then reads as follows:

Theorem 2.6. *Let $\mathcal{V}_1, \mathcal{V}_2$ be any separable Hilbert spaces, let $X, Y \in \mathcal{B}_2(\mathcal{V}_1, \mathcal{V}_2)$ be arbitrary and denote their singular values by $\sigma_j(X), \sigma_j(Y)$, respectively. Then*

$$\langle X, Y \rangle_{\mathcal{B}_2} \leq \sum_{j=1}^n \sigma_j(X) \sigma_j(Y)$$

with equality if and only if the singular vectors can be chosen identically.

The statement is well known for matrices but, surprisingly, the infinite dimensional version is nowhere to be found in the standard literature on operator theory, and we have also not been able to locate it any scientific publication. For that reason, we include a proof in the appendix.

Proposition 2.7. *Let $\mathcal{V}_1, \mathcal{V}_2$ be any separable Hilbert spaces. Suppose that f is a permutation and sign invariant $[0, \infty]$ -valued functional on \mathbb{R}^n and that $F(X) = f(\sigma(X))$. Then*

$$F^\circ(Y) = f^\circ(\sigma(Y)).$$

In particular, this identity holds for all matrices.

Proof. Since $F^\circ(Y) = \sup_X -f(\sigma(X)) - \frac{1}{2} \|X - Y\|_{\mathcal{B}_2}^2$, von-Neumann's inequality implies that the supremum is attained for an X that shares singular vectors with Y . Hence

$$F^\circ(Y) = \sup_{\gamma_1 \geq \gamma_2 \geq \dots} -f(\gamma) - \frac{1}{2} \|\gamma - \sigma(Y)\|_2^2.$$

Due to the permutation and sign invariance of f , we can drop the restrictions on γ and so

$$F^\circ(Y) = \sup_{\gamma} -f(\gamma) - \frac{1}{2} \|\gamma - \sigma(Y)\|_2^2 = f^\circ(\sigma(Y)).$$

□

It is now easy to determine the \mathcal{S} -transform of the rank-functional on matrices.

Example 2.8. *Recall (15). By combining Propositions 2.5, 2.7 with (17) and (18) we immediately get that*

$$\text{rank}^\circ(Y) = \sum_{j=1}^n \max \left\{ \frac{-\sigma_j(Y)^2}{2}, -1 \right\} \quad (20)$$

and

$$\text{rank}^{\circ\circ}(X) = \sum_{j=1}^n 1 - \left(\max \left\{ 1 - \frac{\sigma_j(X)}{\sqrt{2}}, 0 \right\} \right)^2. \quad (21)$$

More generally, given two separable infinite-dimensional Hilbert spaces $\mathcal{V}_1, \mathcal{V}_2$ and constants $\tau, \nu > 0$, let \mathcal{V} be the vector space $\mathcal{B}_2(\mathcal{V}_1, \mathcal{V}_2)$ equipped with the reweighted norm $\|X\|_{\mathcal{V}} = \nu \|X\|_{\mathcal{B}_2}$. Setting $f(X) = \tau^2 \text{rank}(X)$, we have

$$\mathcal{S}_{\mathcal{V}}(f)(Y) = \sum_{j=1}^{\infty} \max \left\{ \frac{-\nu^2 \sigma_j(Y)^2}{2}, -\tau^2 \right\} \quad (22)$$

and

$$\mathcal{S}_{\mathcal{V}}(\mathcal{S}_{\mathcal{V}}(f))(X) = \sum_{j=1}^{\infty} \tau^2 - \left(\max \left\{ \tau - \frac{\nu \sigma_j(X)}{\sqrt{2}}, 0 \right\} \right)^2. \quad (23)$$

For applications where one minimizes a functional over a certain linear subspace of $\mathbb{M}_{m,n}$, the unweighted Frobenius norm is not always the most optimal choice, as the next example highlights.

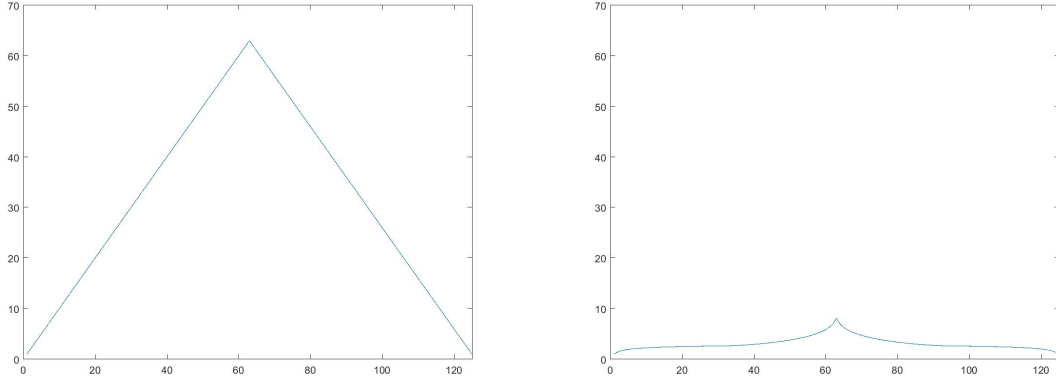


Figure 5: Left; the weight appearing in (24) for $n = 63$. Right; corresponding weight for (28).

Example 2.9. Fix $n \in \mathbb{N}$ and let $H_f \in \mathbb{M}_{n,n}$ be the Hankel matrix generated by the sequence $f = (f_1, \dots, f_{n-1}) \in \mathbb{C}^{2n-1}$. If one is interested in minimizing the rank of a Hankel matrix H_f while at the same time not deviating far from some measurement $d \in \mathbb{C}^{2n-1}$, as is frequent in frequency estimation [1], one option is to minimize the functional $\text{rank}(X) + \frac{1}{2} \|X - H_d\|_F^2$ over the set of Hankel matrices, (we consider minimization over subspaces in more detail in Part II). Setting $X = H_f$, the quadratic term $\|X - H_d\|_F^2$ corresponds to a weighted misfit term of the form

$$\|H_f - H_d\|_F^2 = \sum_{j=1}^{2n-1} (n - |j - n|) |f_j - d_j|^2, \quad (24)$$

(see Figure 5, left) which is not the most natural quantity to minimize, as has been observed by many authors (e.g. [16]).

To partially remedy the issues highlighted in the previous example, we include a few results on how to compute \mathcal{S} -transforms with respect to certain weighted spaces of matrices. Given $W \in \mathbb{M}_{m,n}$ with (strictly) positive entries, we let $\mathbb{M}_{m,n}^W$ be the Hilbert space obtained by introducing the norm

$$\|X\|_W^2 = \sum_{i,j} w_{i,j} |x_{i,j}|^2,$$

where e.g. $w_{i,j}$ are the entries of W . In case $W = \mathbf{1}$, i.e. W is equal to one componentwise, we will simply write $\mathbb{M}_{m,n}$ as earlier. Suppose now that we are interested in computing $\mathcal{S}_{\mathbb{M}_{m,n}^W}(f)$, where f is such that $\mathcal{S}_{\mathbb{M}_{m,n}}(f)$ has an explicit expression. In general, this will only be possible if W is a direct tensor, i.e. of the form

$$w_{i,j} = u_i v_j \quad (25)$$

where u and v are sequences of length m and n respectively. The following examples and proposition show how to do this. A linear operator between two spaces that is bijective and isometric will be referred to as unitary.

Example 2.10. Under the assumption (25), note that

$$X \mapsto I_{\sqrt{v}} X I_{\sqrt{u}}$$

is unitary between $\mathbb{M}_{m,n}^W$ and $\mathbb{M}_{m,n}$, where e.g. $I_{\sqrt{u}}$ is a diagonal matrix with $\sqrt{u} = (\sqrt{u_j})_{j=1}^n$. Also note that $I_{\sqrt{v}} : \mathbb{M}_{m,1}^v \rightarrow \mathbb{C}^m$ and $I_{\sqrt{u}} : \mathbb{C}^n \rightarrow \mathbb{M}_{n,1}^{1/u}$ are unitary, where $1/u$ refers to componentwise division. The space $\mathbb{M}_{n,1}^{1/u}$ is of course the same as \mathbb{C}^n as a vector space, but with a different norm. In fact, if e_1, \dots, e_n denotes the canonical basis in \mathbb{C}^n , we have that $u_j = \sqrt{u_j} e_j$, $j = 1, \dots, n$ defines an orthonormal basis in $\mathbb{M}_{n,1}^{1/u}$. Each matrix $X = (x_{i,j}) \in \mathbb{M}_{m,n}^W$ defines an operator $X \in \mathcal{B}_2(\mathbb{M}_{n,1}^{1/u}, \mathbb{M}_{m,1}^v)$ in the obvious way, i.e. $(Xy)_i = \sum_j x_{i,j} y_j$. It is easy to see that

$$\|X\|_{\mathcal{B}_2(\mathbb{M}_{n,1}^{1/u}, \mathbb{M}_{m,1}^v)}^2 = \sum_{j=1}^n \|Xu_j\|_{\mathbb{M}_{m,1}^v}^2 = \sum_{j=1}^n \sum_{i=1}^m u_j v_i |x_{i,j}|^2 = \|X\|_{\mathbb{M}_{m,n}^W}^2.$$

Proposition 2.11. Let $\mathcal{V}_1, \tilde{\mathcal{V}}_1, \mathcal{V}_2$ and $\tilde{\mathcal{V}}_2$ be separable Hilbert spaces, let $I_1 : \mathcal{V}_1 \rightarrow \tilde{\mathcal{V}}_1$ be unitary and let $I_2 : \tilde{\mathcal{V}}_2 \rightarrow \mathcal{V}_2$ be unitary. Then the induced map $\mathcal{J} : \mathcal{B}_2(\tilde{\mathcal{V}}_1, \tilde{\mathcal{V}}_2) \rightarrow \mathcal{B}_2(\mathcal{V}_1, \mathcal{V}_2)$ given by $\mathcal{J}(X) = I_2 X I_1$ is unitary.

Moreover, let f be an $[0, \infty]$ -valued functional on $\mathcal{B}_2(\tilde{\mathcal{V}}_1, \tilde{\mathcal{V}}_2)$. Then

$$\mathcal{S}_{\mathcal{B}_2(\tilde{\mathcal{V}}_1, \tilde{\mathcal{V}}_2)}(f)(Y) = \mathcal{S}_{\mathcal{B}_2(\mathcal{V}_1, \mathcal{V}_2)}(f \circ \mathcal{J}^{-1})(\mathcal{J}(Y))$$

and

$$(\mathcal{S}_{\mathcal{B}_2(\tilde{\mathcal{V}}_1, \tilde{\mathcal{V}}_2)}(f))^2(X) = (\mathcal{S}_{\mathcal{B}_2(\mathcal{V}_1, \mathcal{V}_2)}(f \circ \mathcal{J}^{-1}))^2(\mathcal{J}(X))$$

Proof. The first statement is immediate by the definition of the Hilbert-Schmidt norm. The first identity follows from the calculation

$$\begin{aligned} \mathcal{S}_{\mathcal{B}_2(\tilde{\mathcal{V}}_1, \tilde{\mathcal{V}}_2)}(f)(Y) &= \sup_{X \in \mathcal{B}_2(\tilde{\mathcal{V}}_1, \tilde{\mathcal{V}}_2)} -f(X) - \frac{1}{2} \|X - Y\|_{\mathcal{B}_2(\tilde{\mathcal{V}}_1, \tilde{\mathcal{V}}_2)}^2 = \\ &= \sup_{X \in \mathcal{B}_2(\tilde{\mathcal{V}}_1, \tilde{\mathcal{V}}_2)} -f(\mathcal{J}^{-1}(\mathcal{J}X)) - \frac{1}{2} \|\mathcal{J}(X - Y)\|_{\mathcal{B}_2(\mathcal{V}_1, \mathcal{V}_2)}^2 = \\ &= \sup_{X \in \mathcal{B}_2(\mathcal{V}_1, \mathcal{V}_2)} -f(\mathcal{J}^{-1}(Z)) - \frac{1}{2} \|Z - \mathcal{J}Y\|_{\mathcal{B}_2(\mathcal{V}_1, \mathcal{V}_2)}^2 = \mathcal{S}_{\mathcal{B}_2(\mathcal{V}_1, \mathcal{V}_2)}(f \circ \mathcal{J}^{-1})(\mathcal{J}Y), \end{aligned}$$

and the latter is a consequence of applying the former twice. \square

Example 2.12. We continue Example 2.10. Set $\mathcal{V}_1 = \mathbb{C}^n$, $\tilde{\mathcal{V}}_1 = \mathbb{M}_{n,1}^{1/u}$, $I_1 = I_{\sqrt{u}}$, $\mathcal{V}_2 = \mathbb{C}^m$, $\tilde{\mathcal{V}}_2 = \mathbb{M}_{m,1}^v$, $I_2 = I_{\sqrt{v}}$ and $f = \text{rank}$. Note that $f \circ \mathcal{J}^{-1} = f$ since left or right multiplication with invertible diagonal matrices do not change the rank. By proposition 2.11 and Example 2.8 we conclude that

$$\mathcal{S}_{\mathbb{M}_{m,n}^W}(\text{rank})(Y) = \mathcal{S}_{\mathbb{M}_{m,n}}(\text{rank})(\mathcal{J}(Y)) = \sum_{j=1}^n \max \left\{ \frac{-\sigma_j(I_{\sqrt{v}} Y I_{\sqrt{u}})^2}{2}, -1 \right\} \quad (26)$$

and

$$\mathcal{S}_{\mathbb{M}_{m,n}^W}^2(\text{rank})(X) = \mathcal{S}_{\mathbb{M}_{m,n}}^2(\text{rank})(\mathcal{J}(X)) = \sum_{j=1}^n 1 - \left(\max\left\{1 - \frac{\sigma_j(I_{\sqrt{v}}X I_{\sqrt{u}})}{\sqrt{2}}, 0\right\} \right)^2, \quad (27)$$

generalizing (20) and (21).

Example 2.13. Continuing example 2.9 we consider minimization of the functional

$$\text{rank}(X) + \frac{1}{2} \|X - H_d\|_W^2$$

over the set of Hankel matrices, where we assume that $n = 2k - 1$ is odd and that $w_{i,j} = u_i u_j$ with $u_i = \frac{1}{\sqrt{k-|i-k|}}$. By the above theory the convex hull is given by

$$\sum_{j=1}^n 1 - \left(\max\left\{1 - \frac{\sigma_j(I_{\sqrt{u}}X I_{\sqrt{u}})}{\sqrt{2}}, 0\right\} \right)^2 + \frac{1}{2} \|X - H_d\|_W^2.$$

Inserting $X = H_f$ in the quadratic term gives

$$\|H_f - H_d\|_W^2 = \sum_{j=1}^{2n-1} w_j |f_j - d_j|^2, \quad (28)$$

where w_j is depicted in Figure 5, right. Compared with (24), this weight is clearly much closer to a uniform flat weight (both weights (24) and (28) start and end with the weight 1, so the scaling in Figure 5 is fair). What the optimal choice of u would be in order to yield as flat a weight as possible, is to our knowledge an open question.

We now change focus and take a look at functionals forcing a predetermined amount of non-zero terms.

Example 2.14. On \mathbb{R}^d define $F_K(x) = \begin{cases} 0 & \|x\|_0 \leq K, \\ \infty & \text{else.} \end{cases}$ and define \tilde{x} to be the vector x resorted so that $(|\tilde{x}_j|)_{j=1}^d$ is a decreasing sequence. Then

$$F_K^\circ(y) = \sum_{j=K+1}^d -\frac{1}{2} |\tilde{y}_j|^2.$$

To see this, note that $F_K^\circ(y) = \sup_x -F_K(x) - \frac{1}{2} \sum_{j=1}^d (x_j - y_j)^2$, and it is clear that the optimal value of x_j is y_j if $|y_j|$ is among the K greatest, and zero else.

The computation of $F_K^{\circ\circ}$ is more involved. The expression is

$$F_K^{\circ\circ}(x) = \frac{1}{2k_*} \left(\sum_{j>K-k_*} |\tilde{x}_j| \right)^2 - \frac{1}{2} \sum_{j>K-k_*} |\tilde{x}_j|^2$$

where k_* is a particular number between 1 and K , we refer to [2] for the details.

Example 2.15. Letting F_K be as above we can now use Proposition 2.7 to see that $F_K(\sigma(X))$ has \mathcal{S} -transform $F_K^{\circ\circ}(\sigma(X))$, we refer to [2] or [18] for more information on this particular functional. The latter reference investigates the present example and Example 2.8 in a more general framework, looking at functionals of the form $g(\text{rank}(X))$ where g is a “convex” non-decreasing functional on the natural numbers (see eq. (5) in [18] for a precise definition). They derive a feasible algorithm for computing $(g(\text{rank}(X)))^{\circ\circ}$, which in their nomenclature is denoted $\mathcal{R}_g(X)$ (see eq. (19)). It is easy to see that the same method can be adapted to also deal with functionals on \mathbb{R}^n of the form $x \mapsto g(|\tilde{x}|)$, where \tilde{x} is as in Example 2.14.

We now look at functionals on eigenvalues rather than singular values, and in particular functionals that avoid negative eigenvalues. Let \mathcal{V} be any separable Hilbert space, denote by $\mathbb{H}(\mathcal{V}) \in \mathcal{B}_2(\mathcal{V}, \mathcal{V})$ the space of self-adjoint (Hermitian) operators, and let $\lambda(X)$ denote the eigenvalues of a given $X \in \mathbb{H}(\mathcal{V})$. In case \mathcal{V} is of finite dimension n , so that $\mathcal{B}_2(\mathcal{V}, \mathcal{V}) \simeq \mathbb{M}_{n,n}$, we simply write \mathbb{H}_n . By a variation of the proof of Proposition 2.7 we also have

Proposition 2.16. Let \mathcal{V} be a separable Hilbert space. Suppose that f is a permutation invariant functional on $\mathbb{R}^{\dim \mathcal{V}}$ and that $F : \mathbb{H}(\mathcal{V}) \rightarrow \mathbb{R}$ is given by $F(X) = f(\lambda(X))$. Then

$$F^{\circ}(Y) = f^{\circ}(\lambda(Y)).$$

Suppose we are interested in positive matrices with low rank. In analogy with the previous developments, this calls for an investigation of the following functional.

Example 2.17. Set $\mathcal{V} = \mathbb{R}$ with norm $\|x\|_{\mathcal{V}}^2 = \nu^2|x|^2$ and set

$$f(x) = \tau^2 \chi_{(0,\infty)}(x) + \infty \chi_{(-\infty,0)}$$

where $\tau, \nu > 0$ are fixed (see Figure 2.2). By a variation of the calculations in Example 2.4, we have

$$f^{\circ}(y) = - \left(\min \left\{ \frac{\nu y}{\sqrt{2}}, \tau \right\} \right)^2$$

and

$$f^{\circ\circ}(x) = \tau^2 - \left(\max \left\{ \tau - \frac{\nu x}{\sqrt{2}}, 0 \right\} \right)^2 + \infty \chi_{(-\infty,0)}(x) \quad (29)$$

Example 2.18. Let f be as above with $\tau = \nu = 1$ and let $\mathbb{P}_n \subset \mathbb{H}_n$ be the set of positive matrices. Define F on \mathbb{H}_n by

$$F(X) = \sum_{j=1}^n f(\lambda_j(X)) = \text{rank}(X) + \infty \chi_{\{X \notin \mathbb{P}_n\}}(X).$$

Then

$$F^{\circ\circ}(X) = \sum_{j=1}^n f^{\circ\circ}(\lambda_j(X)) = \sum_{j=1}^n 1 - \left(\max \left\{ 1 - \frac{\lambda_j(X)}{\sqrt{2}}, 0 \right\} \right)^2 + \infty \chi_{\{X \notin \mathbb{P}_n\}}(X),$$

as follows by combining Proposition 2.5, 2.16 with Example 2.17.

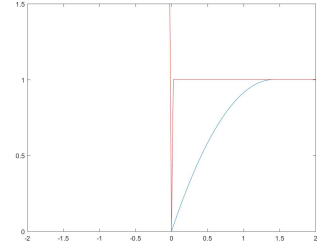


Figure 6: Illustration of f (red) and $f^{\circ\circ}$ (blue) in Example 2.17, with $\tau = \nu = 1$.

Finally, suppose we want to have at most K positive eigenvalues and no negative ones.

Example 2.19. On \mathbb{R}^d define $F_K^+(x) = \begin{cases} 0 & \|x\|_0 \leq K \text{ and } x \geq 0, \\ \infty & \text{else.} \end{cases}$. By a refinement of Example 2.14 we have

$$(F_K^+)^{\circ}(y) = \frac{1}{2} \left(\sum_{j=1}^K |\max(\tilde{y}_j, 0)|^2 - \|y\|^2 \right).$$

We omit a computation of $(F_K^+)^{\circ\circ}$, partially because we shall show in the coming section that it is not needed for the evaluation of the proximal operator, and partially because it is a rather involved computation that we will perform in detail elsewhere.

2.3. Weak lower semi-continuity and finer properties of l.s.c. convex hulls

This section is a natural continuation of Section 2.1, in which we state general properties that \mathcal{S} -transforms obey. A functional f on a separable Hilbert space \mathcal{V} is called weakly l.s.c. if it is lower semi-continuous with respect to the weak topology. We remind the reader that all finite dimensional topological vector spaces have the same topology (Exc. 18, Ch. IV.1 [12]), and hence there is no difference between weakly l.s.c. functionals and standard l.s.c. functionals in this case. Also for convex proper functionals in the infinite dimensional case there is no difference (Theorem 9.1 [5]). We now provide two examples of weakly l.s.c. functionals in the ∞ -dimensional case, that are of interest to the setting of this paper.

Example 2.20. In $\ell^2(\mathbb{N})$ the counting functional $\|x\|_0 = \#\{k \in \mathbb{N} : x_k \neq 0\}$ is weakly l.s.c.

Proof. We need to check that the preimage of an open interval of the form (k, ∞) is open in the weak topology. Let x be such that $K = \|x\|_0 > k$, and let $j_1 < j_2 < \dots < j_K$ be the indices for which $x_{j_j} \neq 0$. Set

$$V = \cap_{m=1}^K \{y \in \ell^2(\mathbb{N}) : |y_{j_m} - x_{j_m}| < |x_{j_m}|/2\}$$

which is open in the weak topology. Clearly $x \in V$ and $\|y\|_0 \geq K$ for all $y \in V$, which proves the claim. \square

Let $\mathcal{V}_1, \mathcal{V}_2$ be separable Hilbert spaces and recall that $\mathcal{B}_2(\mathcal{V}_1, \mathcal{V}_2)$ denotes all Hilbert-Schmidt operators with the Hilbert-Schmidt norm. We remark that in case $\mathcal{V}_1 = \mathbb{C}^n$ and $\mathcal{V}_2 = \mathbb{C}^m$ with the canonical norms, then $\mathcal{B}_2(\mathcal{V}_1, \mathcal{V}_2)$ equals $\mathbb{M}_{m,n}$ with the Frobenius norm.

Example 2.21. The rank functional is weakly l.s.c. on $\mathcal{B}_2(\mathcal{V}_1, \mathcal{V}_2)$.

Proof. We only focus on the infinite dimensional case. Let $X \in \mathcal{B}_2(\mathcal{V}_1, \mathcal{V}_2)$ have $\text{rank}(X) \geq K$. As before we need to produce an open set V including X such that $\text{rank}(Y) \geq K$ for all $Y \in V$. By the polar decomposition of compact operators (9) we may pick an orthonormal basis $(v_k)_{k=1}^{\infty}$ for \mathcal{V}_1 and an orthonormal sequence $(u_k)_{k=1}^{\infty}$ such that

$$X = \sum_{k=1}^{\infty} \sigma_k u_k \otimes v_k$$

where σ_k are the singular values ordered decreasingly. Clearly $\sigma_K > 0$ and $(X, u_k \otimes v_j) = \sigma_k \delta_0(j - k)$ where δ_0 is the characteristic function of $\{0\}$ on \mathbb{R} . Define

$$V = \cap_{j,k=1}^K \{Y \in \mathcal{B}_2(\mathcal{V}_1, \mathcal{V}_2) : |(Y, u_k \otimes v_j) - (X, u_k \otimes v_j)| < \epsilon\},$$

where ϵ will be determined later. Let ι_1 and ι_2 be the canonical inclusions (i.e. the operator that sends a vector into itself) from $\text{Span}\{u_k\}_{k=1}^K$ and $\text{Span}\{v_k\}_{k=1}^K$ into \mathcal{V}_1 and \mathcal{V}_2 respectively, and note that ι_2^* then acts as the orthogonal projection onto $\text{Span}\{v_k\}_{k=1}^K$. Pick $Y \in V$ and note that

$$\text{rank}(\iota_2^* Y \iota_1) \leq \text{rank}(Y). \quad (30)$$

Moreover $\iota_2^* Y \iota_1$ is an operator on finite dimensional spaces which in the bases $\{u_k\}_{k=1}^K$ and $\{v_k\}_{k=1}^K$ have the matrix representation

$$(y_{j,k})_{j,k=1}^K = ((u_j, Y v_k))_{j,k=1}^K = ((Y, u_j \otimes v_k))_{j,k=1}^K.$$

It follows that, upon choosing ϵ sufficiently small, we can make this matrix arbitrarily close (pointwise) to the diagonal matrix with σ_k on the diagonal. By basic linear algebra it follows that $\text{rank}(\iota_2^* Y \iota_1) = K$ for ϵ sufficiently small, independent of Y . Combining this with (30) we conclude that $\text{rank}(Y) \geq K$, which was to be shown. \square

We now move on to discuss properties of convex hulls under the assumption of weak l.s.c. We remind the reader that a functional g is coercive if and only if its (lower) level sets are bounded, (see e.g. Proposition 11.11 [5]). Note that convex hulls of the type $f^{\circ\circ}(x) + \frac{1}{2}\|x - d\|_{\mathcal{V}}^2$ (for positive f) always are coercive, by virtue of Proposition 2.2 and the quadratic term. The results below are of key importance to Part III, where we treat the use of the \mathcal{S} -transform for problems such as (2), where the l.s.c. convex envelope is not explicitly computable. These results rely on a neat fact concerning weakly l.s.c. convex envelopes which does not seem to have made its way into the modern literature on the subject, given below. We recall Proposition 2.1 that g^{**} is the l.s.c. convex envelope of a given functional g . A function $f(t)$ is called affine if it is of the form $at + b$ with $a, b \in \mathbb{R}$.

Theorem 2.22. *Let g be a weakly l.s.c. functional on a separable Hilbert space \mathcal{V} such that g^{**} is coercive. Given any $x \in \mathcal{V}$, we either have $g(x) = g^{**}(x)$ or there exists a unit vector ν and $t_0 > 0$ such that the function $h(t) = g^{**}(x_0 + t\nu)$ is affine on $(-t_0, t_0)$.*

We remark that both statements may hold simultaneously. The theorem should be considered in the light of the (non-trivial) fact that we may have $g^{**}(x) > g(x)$ and yet that the subdifferential of g^{**} is empty. To not disrupt the flow of ideas we give the proof of Theorem 2.22, which is based on an extension of the Milman theorem by Arne Brøndsted [8], in the appendix. Denote by G the set of global minimizers of g and by G^{**} the set of global minimizers of g^{**} .

Corollary 2.23. *Let g be a weakly l.s.c. functional on a separable Hilbert space \mathcal{V} such that g is coercive. Then G^{**} is a closed bounded convex set containing G . Letting G_{ext}^{**} denote the extremal points of G^{**} , we also have that $G_{ext}^{**} \subset G$. Finally, the closed convex hull of G_{ext}^{**} equals G^{**} .*

Proof. The convexity of G^{**} and the inclusion $G \subset G^{**}$ are immediate. The boundedness of G^{**} follows since g^{**} is coercive. Let x be in the closure of G^{**} , and let c be the value of the global minimum. Then $g^{**}(x) \leq c$ follows by l.s.c., and the reverse inequality is obvious from the fact that c is a global minimum. It follows that $x \in G^{**}$ and hence G^{**} is closed.

The existence of points in G_{ext}^{**} and the statement concerning the closed convex hull are now immediate consequences of the Krein-Milman theorem (see e.g. [12]) and the fact that bounded closed convex sets are weakly compact in separable Hilbert spaces (Theorem 3.33, [5]). It remains to prove that $G_{ext}^{**} \subset G$. Let $x_0 \in G_{ext}^{**}$ suppose $x_0 \notin G$. Then Theorem 2.22 implies the existence of a direction ν on which g^{**} is constant near x_0 , contradicting that x_0 is an extremal point. \square

Corollary 2.24. *Let f be a weakly l.s.c. $[0, \infty]$ -valued functional on a separable Hilbert space \mathcal{V} and set $\Omega = \{x \in \mathcal{V} : f(x) > f^{\circ\circ}(x)\}$. For each $x_0 \in \Omega$ there exists a unit vector ν and $t_0 > 0$ such that the function $h(t) = f^{\circ\circ}(x_0 + t\nu)$ has second derivative -1 on $(-t_0, t_0)$.*

Proof. Set $g(x) = f(x) + \frac{1}{2}\|x\|_{\mathcal{V}}^2$. By Theorem 2.3 we have

$$f^{\circ\circ}(x) + \frac{1}{2}\|x\|_{\mathcal{V}}^2 = g^{**}(x),$$

by which it is immediate that g^{**} is coercive, (since $f^{\circ\circ} \geq 0$ by Proposition 2.2). It also follows that

$$\Omega = \{x \in \mathcal{V} : g(x) > g^{**}(x)\} \quad (31)$$

and hence, for $x \in \Omega$, Theorem 2.22 implies that a unit vector ν exists such that $t \mapsto f^{\circ\circ}(x + t\nu)$ equals an affine function minus $\frac{1}{2}t^2$ in a neighborhood of $t = 0$. The corollary follows. \square

3. Part II; applications when explicit formulas are available

3.1. Minimization over convex subsets

Suppose \mathcal{M} is a closed convex subset of \mathcal{V} and that we wish to solve

$$\arg \min_{x \in \mathcal{M}} f(x) + \frac{1}{2}\|x - d\|_{\mathcal{V}}^2, \quad (32)$$

which we replace by

$$\arg \min_{x \in \mathcal{M}} f^{\circ\circ}(x) + \frac{1}{2}\|x - d\|_{\mathcal{V}}^2 \quad (33)$$

to obtain a convex problem. As highlighted by Figure 2, this is a different problem with possibly a different answer. However, as seen in Figure 7 the two problems may have the same answer, and it is easy to see that this happens precisely when

$$f(\hat{x}) = f^{\circ\circ}(\hat{x}) \quad (34)$$

holds for all solutions \hat{x} of (33) (since $f \geq f^{\circ\circ}$ by Theorem 2.3) which is easily verified if a concrete expression for $f^{\circ\circ}$ is available. The rationale behind solving (33) as opposed to (32) is pragmatical; since the latter is convex the solution may be found using convex optimization routines. For example, as shown in [3], it is often possible to solve this problem *without*

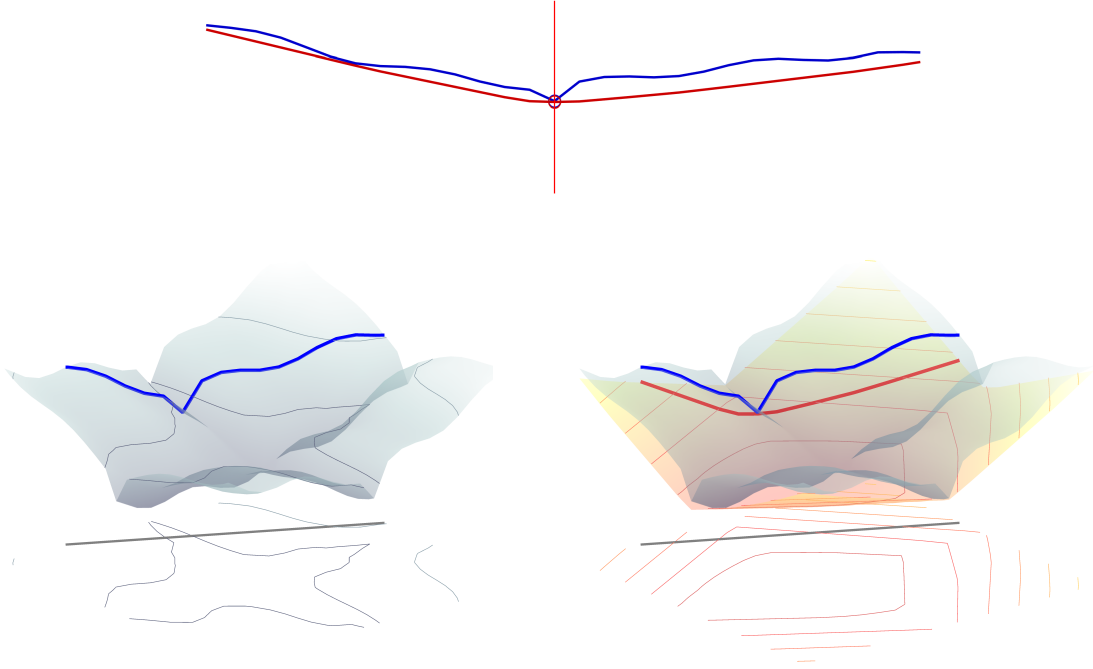


Figure 7: Same setup as in Figure 2, but with a different subspace. Note that the minima of both problems (32) and (33) coincide.

knowing $f^{\circ\circ}$, using dual ascent. However, since the functional (33) is not strictly convex, the convergence results are not as strong as one would desire.

To improve the situation, pick a small number $\epsilon > 0$ and set $f_\epsilon = (1 + \epsilon)f$. Problem (32) is then equivalent with

$$\arg \min_{x \in \mathcal{M}} f_\epsilon(x) + \frac{1 + \epsilon}{2} \|x - d\|_{\mathcal{V}}^2, \quad (35)$$

which we may replace by

$$\arg \min_{x \in \mathcal{M}} f_\epsilon^{\circ\circ}(x) + \frac{1 + \epsilon}{2} \|x - d\|_{\mathcal{V}}^2. \quad (36)$$

The following proposition summarizes the main results of this section, (see [5] for explanation of the terminology).

Proposition 3.1. *Let f be a $[0, \infty]$ -valued functional on a separable Hilbert space \mathcal{V} , and let \mathcal{M} be a closed convex subset such that f is not identically ∞ on \mathcal{M} . Given $\epsilon > 0$, the functional in (36) is strongly convex and supercoercive. The solution is thus a unique point \hat{x} , which solves (32) whenever $f_\epsilon^{\circ\circ}(\hat{x}) = (1 + \epsilon)f(\hat{x})$.*

Proof. By Theorem 2.3, the functional $f_\epsilon^{\circ\circ}(x) + \frac{1}{2}\|x - d\|_{\mathcal{V}}^2$ is l.s.c. convex (but not necessarily strictly convex), and hence the functional in (36) is l.s.c. and strongly convex. Supercoercivity is obvious due to the term $\frac{1}{2}\|x - d\|_{\mathcal{V}}^2$ since $f_\epsilon^{\circ\circ} \geq 0$. Corollary 11.16 in [5] applied to the functional $f_\epsilon^{\circ\circ}(x) + \frac{1+\epsilon}{2}\|x - d\|_{\mathcal{V}}^2 + \iota_{\mathcal{M}}$, where $\iota_{\mathcal{M}}(x) = 0$ for all $x \in \mathcal{M}$ and ∞ elsewhere, shows that (36) has a unique minimizer \hat{x} . Since $f_\epsilon \geq f_\epsilon^{\circ\circ}$, it follows that \hat{x} solves (35) under the assumption that $f_\epsilon^{\circ\circ}(\hat{x}) = f_\epsilon(\hat{x})$. By the equivalence of (32) and (35), the proof is complete. \square

Several algorithms can be used to solve (36). For example, if \mathcal{M} is a closed linear subspace we may let \mathcal{H} be some linear operator with range equal to \mathcal{M} and solve

$$\arg \min_{x,y} f_\epsilon^{\circ\circ}(x) + \frac{1}{2}\|x - d\|^2 + \frac{\epsilon}{2}\|\mathcal{H}(y) - d\|^2 \quad (37)$$

subject to $x = \mathcal{H}(y)$ using ADMM. This approach considers the Lagrangian

$$\mathcal{L}(x, y, \Lambda) = f_\epsilon^{\circ\circ}(x) + \frac{1}{2}\|x - d\|^2 + \frac{\epsilon}{2}\|\mathcal{H}(y) - d\|^2 + \langle x - \mathcal{H}(y), \Lambda \rangle + \frac{\rho}{2}\|x - \mathcal{H}(y)\|^2 \quad (38)$$

and iteratively seeks minima over x and y separately. The update over x then reduces to computing the proximal operator

$$\mathcal{P}_{\rho, f_\epsilon^{\circ\circ}}(d) = \arg \min_x f_\epsilon^{\circ\circ}(x) + \frac{1+\rho}{2}\|x - \tilde{d}\|^2, \quad (39)$$

with $\tilde{d} = \frac{d + \rho\mathcal{H}(y) - \Lambda}{1+\rho}$. Another possibility for solving (36) is to modify the augmented dual ascent method considered in [3], and a third possibility is to use forward-backward splitting (see e.g. [10]) on

$$y \mapsto \left(f_\epsilon^{\circ\circ}(\mathcal{H}(y)) + \frac{1}{2}\|\mathcal{H}(y) - d\|_{\mathcal{V}}^2 \right) + \frac{\epsilon}{2}\|\mathcal{H}(y) - d\|_{\mathcal{V}}^2.$$

Other attempts at minimizing non-convex functionals include [6, 17, 27], however these articles include assumptions on the functionals that rule out the examples of the present paper, and moreover the connection with convex envelopes is not present. We also mention [1], where a new fixed point algorithm for solving (32) is given in the case of matrices and rank minimization, with applications to multidimensional frequency estimation. Finally under certain assumptions on f , stationary points of the non-convex problem (32) can be solved by so called forward-backward splitting, see e.g. [4, 7].

3.2. Minimization with additional priors

We now consider another example, let c be a convex functional on \mathcal{V} incorporating prior information known about the problem in question, and suppose we wish to minimize

$$\arg \min_{x \in \mathcal{M}} f(x) + \frac{1}{2}\|x - d\|_{\mathcal{V}}^2 + \lambda c(x), \quad (40)$$

where λ is a tradeoff parameter. This example actually generalizes the previous because we can take as c the indicator function of \mathcal{M} , i.e. $c = \iota_{\mathcal{M}}$. Assuming instead that c is a smooth everywhere finite functional, (40) can be solved with the forward-backward splitting or with ADMM. The latter case is considered in Section 4 of [18], as an application to structure from motion. The same paper also contain several other interesting examples, e.g. system identification.

3.3. The proximal operator

We now study the problem of computing the proximal operator of $f^{\circ\circ}(x) + \frac{1}{2}\|x - d\|_{\mathcal{V}}^2$, i.e.

$$\mathcal{P}_{\rho, f^{\circ\circ}}(d) = \arg \min_x f^{\circ\circ}(x) + \frac{1+\rho}{2}\|x - d\|_{\mathcal{V}}^2. \quad (41)$$

Obviously, if one has a concrete expression for $f^{\circ\circ}$ it may be possible to compute (41) directly. If this is not the case, the following proposition, which is a slight alteration of the classical Moreau decomposition ([19] Sec. 2.5), is of help. In fact, even when both options are available, they may lead to different methods for the evaluation of \mathcal{P} . A concrete example of this concerns the functional in Example 2.14, where f° has a very simple expression and $f^{\circ\circ}$ has a very complicated one.

Proposition 3.2.

$$\mathcal{P}_{\rho, f^{\circ\circ}}(d) = \frac{(1+\rho)d - \mathcal{P}_{1/\rho, f^{\circ}}(d)}{\rho}$$

Proof. We have

$$\begin{aligned} \mathcal{P}_{\rho, f^{\circ\circ}}(d) &= \arg \min_x f^{\circ\circ}(x) + \frac{1+\rho}{2}\|x - d\|^2 = \\ &= \arg \min_x \max_y -f^{\circ}(y) - \frac{1}{2}\|x - y\|^2 + \frac{1+\rho}{2}\|x - d\|^2 = \\ &= \arg \min_x \max_y \frac{\rho}{2} \left\| x - \frac{(1+\rho)d - y}{\rho} \right\|^2 - f^{\circ}(y) - \frac{1}{2} \left(1 + \frac{1}{\rho} \right) \|y - d\|^2 \end{aligned}$$

By Sion's minimax theorem [24], we can switch the order of max and min. This shows that the desired x is given by $\frac{(1+\rho)d - y}{\rho}$, where

$$\begin{aligned} y &= \arg \max_y -f^{\circ}(y) - \frac{1}{2} \left(1 + \frac{1}{\rho} \right) \|y - d\|^2 = \\ &= \arg \min_y f^{\circ}(y) + \frac{1}{2} \left(1 + \frac{1}{\rho} \right) \|y - d\|^2 = \mathcal{P}_{1/\rho, f^{\circ}}(d), \end{aligned}$$

which combined gives the sought identity. \square

4. Part III; quadratic terms of the form $\|Ax - d\|^2$

4.1. Motivation and examples

f° is explicitly computable whenever

$$\arg \min_x f(x) + \frac{1}{2}\|x - d\|_{\mathcal{V}}^2 \quad (42)$$

has explicit solutions for all $d \in \mathcal{V}$ and hence the principal use of the techniques developed in the previous sections is related to solving problems with additional constraints. However, there is an abundance of important functionals for which this is not the case, such as

$$\|x\|_0 + \frac{1}{2}\|Ax - d\|_2^2, \quad x \in \mathbb{C}^n \quad (43)$$

where A is some fixed $m \times n$ matrix, and here the key objective is often simply finding the global minimizer. The remainder of the paper is devoted to the study of such cases.

We henceforth consider

$$f(x) + \frac{1}{2} \|Ax - d\|_{\mathcal{W}}^2, \quad x \in \mathcal{V} \quad (44)$$

where \mathcal{V}, \mathcal{W} are possibly different (separable) Hilbert spaces and $A : \mathcal{V} \rightarrow \mathcal{W}$ is linear. We point out that, in case A is bijective and bounded, we may introduce a new Hilbert space $\tilde{\mathcal{V}}$, which equals \mathcal{V} as a vector space but with the new norm $\|x\|_{\tilde{\mathcal{V}}} = \|Ax\|_{\mathcal{W}}$, and then “compute” the l.s.c. convex envelope of (43) by applying $S_{\tilde{\mathcal{V}}}$ twice to f . In case A is not bijective, $\tilde{\mathcal{V}}$ is only a semi-normed space which may not be complete, but we could still develop a theory similar to that in Part I. However, the problem arise since $S_{\tilde{\mathcal{V}}}(f)$ usually has no explicit formula, and hence the theory becomes vacuous. Moreover, when A has a kernel, f is bounded and $f(0) = 0$, it is easy to see that $S_{\tilde{\mathcal{V}}}(S_{\tilde{\mathcal{V}}}(f))(x) = 0$ for all x in the kernel of A , and hence the convex envelope is not a desirable functional for solving e.g. (43).

Henceforth, we will assume that f an $[0, \infty]$ -valued functional such that $S_{\mathcal{V}}(f) = f^\circ$ is computable, and focus on computing (explicit) approximations of $f(x) + \frac{1}{2} \|Ax - d\|_{\mathcal{W}}^2$. The remaining theory is split in two cases, either A is estimated from above or from below. We begin by providing a natural example where both alternatives are possible.

Example 4.1. Let $W \in \mathbb{M}_{m,n}$ be strictly positive and recall that $\mathbb{M}_{m,n}^W$ is equipped with the norm

$$\|X\|_W^2 = \sum_{i,j} w_{i,j} |x_{i,j}|^2$$

(see the text preceding Example 2.10). Suppose we are interested in the l.s.c. convex envelope of the non-convex functional

$$\text{rank}(X) + \frac{1}{2} \|X - D\|_W^2. \quad (45)$$

Note that (45) can be written as

$$\text{rank}(X) + \frac{1}{2} \|A(X) - A(D)\|_F^2 \quad (46)$$

where A is the linear operator on $\mathbb{M}_{m,n}$ of pointwise multiplication with $\sqrt{W_{i,j}}$. From Example 2.12 we know that the l.s.c convex envelope has a closed form expression in the special case when W is a direct tensor, but the majority of weights W are clearly not of this form. So we assume that W is not a direct tensor and hence no explicit formula for the l.s.c. convex envelope of (45) is available. We thus have to satisfy with estimates of the desired l.s.c. convex envelope. Setting $\overline{W} = (\sup_{i,j} W_{i,j}) \cdot \mathbf{1}$ and $\underline{W} = (\min_{i,j} W_{i,j}) \cdot \mathbf{1}$, one can then compute

$$S_{\mathbb{M}_{m,n}^{\overline{W}}}(S_{\mathbb{M}_{m,n}^{\overline{W}}}(\text{rank}))(X) + \frac{1}{2} \|X - D\|_W^2, \quad (47)$$

or

$$S_{\mathbb{M}_{m,n}^{\underline{W}}}(S_{\mathbb{M}_{m,n}^{\underline{W}}}(\text{rank}))(X) + \frac{1}{2} \|X - D\|_W^2, \quad (48)$$

where the transforms in the above formula have explicit expressions by e.g. (the finite dimensional version of) formula (23).

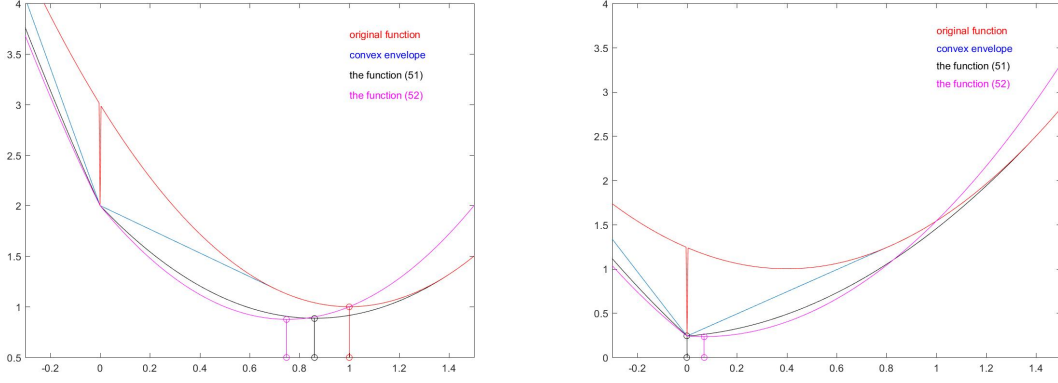


Figure 8: Illustration to Example 4.2. Left graph $A = 2$ and $d = 1$, right graph $A = 1.5$ and $d = 0.4$.

We shall show that (48) is convex whereas (47) is not, and moreover we shall show that the l.s.c. convex envelope sits in between (47) and (48). Henceforth, if we are interested in finding the minimum of (45), we can approximate this by (attempting to) find the minimum of either (47) or (48), and theory for this is provided in the two coming sections.

As a method for “solving” (45), this strategy may seem ad hoc but we remind the reader that minimization of the convex problem

$$\|X\|_* + \frac{1}{2} \|X - D\|_W^2 \quad (49)$$

where $\|X\|_*$ denotes the nuclear norm, has become very popular in recent years, where the rationale behind considering (49) instead of (45) is that the nuclear norm appears as the convex envelope of the rank restricted to the unit ball. Clearly, both (47) and (48) stay closer to the original problem (45) than (49).

Returning the general setting of (44), the two alternatives suggested by (47) and (48) can, upon rescaling, be analyzed assuming either that A is a contraction ($\|A\| \leq 1$) or that A is expansive (i.e. that $\|Ax\| \geq \|x\|$ for all x). We end this section with two concrete toy-examples providing intuition for the two possibilities, which despite their simplicity summarize the general picture. Set $\mathbb{R} = \mathbb{R} \setminus \{0\}$.

Example 4.2. Suppose $\mathcal{V} = \mathbb{R}$ with the norm $\|x\|_{\mathcal{V}}^2 = x^2$ and $A = 2$. Let $d = 1$, suppose we wish to minimize

$$\chi_{\mathbb{R}}(x) + \frac{1}{2} \|2(x - d)\|_{\mathcal{V}}^2 = \chi_{\mathbb{R}}(x) + 2|(x - d)|^2, \quad (50)$$

i.e. the red curve in the left graph of Fig. 8. As is readily seen, the minimum occurs at $x = 1$ which is also the unique minimum of the l.s.c. convex envelope, painted in blue. It differs from (50) in the interval $[-1/\sqrt{2}, 1/\sqrt{2}]$ where it is piecewise linear. A concrete expression can be found by hand or by combining Theorem 2.3 and Example 2.4 with $\tau = 1$ and $\nu = 2$.

However, suppose for the sake of the argument that this minimum, as well as its l.s.c. convex envelope, are impossible to compute analytically. In analogy with (48) we thus replace

(50) by

$$S_{\mathcal{V}}(S_{\mathcal{V}}(\chi_{\mathbb{R}}))(x) + \frac{1}{2} \|2(x - d)\|_{\mathcal{V}}^2, \quad (51)$$

where $S_{\mathcal{V}}(S_{\mathcal{V}}(\chi_{\mathbb{R}}))(x) = 1 - \left(\max\{1 - \frac{|x|}{\sqrt{2}}, 0\}\right)^2$, which follows by (18) with $\tau = 1$ and $\nu = 1$. (51) is depicted in black, and its minimizer is denoted by a black circle. We note that (51) is convex but that the global minimizer has moved. We also include a graph of the expression (49) adapted to this situation, i.e.

$$|x| + \frac{1}{2} \|2(x - d)\|_{\mathcal{V}}^2, \quad (52)$$

showing that in this case the global minimizer moves more (pink curve/circle).

As in Section 3.1, it is not always the case that the global minimizer moves. The right graph shows the same setup but with different values of A and d . In this case the global minimizer of (48) and (50) coincide, but the minimizer of (52) is still different.

Example 4.3. We now reverse the roles. Suppose $\mathcal{V} = \mathbb{R}$ with the norm $\|x\|_{\mathcal{V}}^2 = 16x^2$ and $A = 1/2$. Let $d = 1$ and suppose we wish to minimize

$$\chi_{\mathbb{R}}(x) + \frac{1}{2} \left\| \frac{1}{2}(x - d) \right\|_{\mathcal{V}}^2 = \chi_{\mathbb{R}}(x) + 2|x - d|^2, \quad (53)$$

i.e. the same expression as in (50).

We again suppose that neither the l.s.c. convex envelope nor its minimum is computable explicitly. In analogy with (47) we replace (53) by

$$S_{\mathcal{V}}(S_{\mathcal{V}}(\chi_{\mathbb{R}}))(x) + \frac{1}{2} \left\| \frac{1}{2}(x - d) \right\|_{\mathcal{V}}^2, \quad (54)$$

where $S_{\mathcal{V}}(S_{\mathcal{V}}(\chi_{\mathbb{R}}))(x) = 1 - (\max\{1 - \sqrt{8}|x|, 0\})^2$, which follows by (18) with $\tau = 1$ and $\nu = 4$. This functional, along with its two local minima, is depicted to the left in Figure 9 (black). The remaining functionals are as in Figure 8. We note that (54) is non-convex (but at least continuous) and moreover neither the global nor local minimizers have moved.

To the right we see the same setup but with $d = 0.3$. This shows an important property, namely that the amount of local minimizers of (54) may be fewer than those of (53) (and in real scenarios often is, see [25]). We prove this statement in a more general setting in Section 4.3.

In the coming sections we shall see that the situation in Examples 4.2-4.3 is symptomatic for the general case.

4.2. Expansive A

Again, A is called expansive if $\|Ax\| \geq \|x\|$ for all x , and if the inequality is strict for all x we say that A is strictly expansive. Generalizing the situation in Example 4.2, we suppose that f is a $[0, \infty]$ -valued functional on a separable Hilbert space \mathcal{V} , and that we are interested in the functional

$$g(x) = f(x) + \frac{1}{2} \|Ax - d\|_{\mathcal{V}}^2, \quad (55)$$

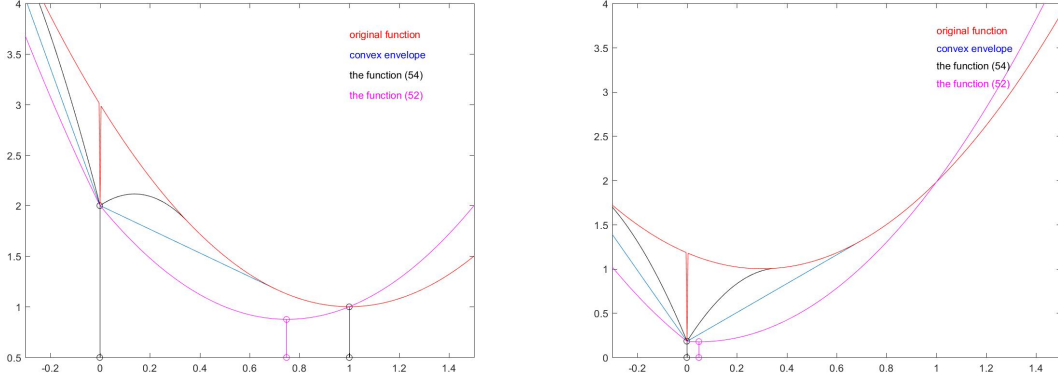


Figure 9: Illustration to Example 4.3. Left $d=1$, right $d=0.3$.

whose l.s.c convex envelope we are unable to compute. We remark that, if $A \in \mathbb{M}_{m,n}$ is not expansive but injective, we may multiply the above expression with $1/\sigma_n$ and replace f with f/σ_n , in order to obtain a functional of the form (55) with expansive A and same minimizers as the one we started with.

Theorem 4.4. *Let f, g be as above and suppose that A is bounded and expansive. Then*

$$h(x) = f^{\circ\circ}(x) + \frac{1}{2} \|Ax - d\|_{\mathcal{V}}^2 \quad (56)$$

is a l.s.c. convex minimizer of g . A minimizer \hat{x} of h is a minimizer of g if $f(\hat{x}) = f^{\circ\circ}(\hat{x})$. If A is strictly expansive, then (56) is strictly convex, in which case it has a unique minimizer.

Proof. Upon expanding $\|Ax - d\|^2 = \|Ax\|^2 + 2\text{Re} \langle Ax, d \rangle + \|d\|^2$, it is easily seen that it suffices to prove the first part of the proposition for $d = 0$. That $h \leq g$ is clear since $f^{\circ\circ} \leq f$ by Theorem 2.3. Define $\langle x, y \rangle_{\mathcal{W}} = \langle Ax, Ay \rangle_{\mathcal{V}} - \langle x, y \rangle_{\mathcal{V}}$ and note that this is a semi-inner product, which is an inner product if A is strictly expansive. In either case, $\|x\|_{\mathcal{W}}^2 := \langle x, x \rangle_{\mathcal{W}}$ is convex (see Chapter I.1 of [12]). It follows that $h(x) = f(x) + \|Ax\|_{\mathcal{V}}^2 = (f(x) + \|x\|_{\mathcal{V}}^2) + \|x\|_{\mathcal{W}}^2$, which by Theorem 2.3 implies that h equals the l.s.c. convex envelope of $f(x) + \frac{1}{2} \|x\|_{\mathcal{V}}^2$ plus the term $\|x\|_{\mathcal{W}}^2$, which obviously is l.s.c. and convex as well. We conclude that h is a l.s.c. convex functional.

Now let d be arbitrary but fixed and suppose that $f(\hat{x}) = f^{\circ\circ}(\hat{x})$ and let $y \in \mathcal{V}$ be arbitrary. Then $g(y) \geq h(y) \geq h(\hat{x}) = g(\hat{x})$, showing that \hat{x} is a global minimizer of g . The statement about strict convexity is immediate since $\|\cdot\|_{\mathcal{W}}^2$ is strictly convex whenever it defines a norm (as opposed to a semi-norm). With this at hand, the existence of a unique minimizer follows by Corollary 11.15 in [5], (supercoercivity of h is obvious as in Proposition 3.1). \square

A drawback of minimizing (56) is that the minimizer most likely not will have a closed form solution (since we assumed that the minimizer of the simpler expression (55) lacks such). Nevertheless, the convexity ensures that your favorite convex optimization routine is likely to converge, albeit, as Example 4.2 and Figure 8 shows, not necessarily to the minimum of (55). Since this paper is primarily concerned with theory for convex envelopes, we do not include a list of potential convex optimization routines suitable for minimizing (56) in the general case.

4.3. Contractive A

We now assume that A is contractive, i.e. $\|A\| \leq 1$, and that f is a $[0, \infty]$ -valued functional. Generalizing the situation in Example 4.3, suppose that we are interested in the functional

$$g(x) = f(x) + \frac{1}{2} \|Ax - d\|_{\mathcal{V}}^2, \quad (57)$$

and that we are able to compute $f^{\circ\circ}$ (or even f° , which often is good enough, see Section 3.3). As in Section 4.2 we remark that, upon multiplying with $1/\|A\| = 1/\sigma_1$, any functional of the form (57) can be rescaled to the same form with A contractive. We remind the reader that g^{**} is the l.s.c. convex envelope of g , where \cdot^* denotes the Fenchel conjugate (Proposition 2.1).

Proposition 4.5. *Let $\|A\| \leq 1$ and let f be a $[0, \infty]$ -valued functional. Then*

$$h(x) = f^{\circ\circ}(x) + \frac{1}{2} \|Ax - d\|_{\mathcal{V}}^2 \quad (58)$$

is an l.s.c. functional satisfying

$$g^{**} \leq h \leq g. \quad (59)$$

Also, it is continuous on the interior of the domain where it is finite.

Proof. As in Theorem 4.4 we may assume that $d = 0$. Setting $\|x\|_{\mathcal{W}}^2 = \|x\|_{\mathcal{V}}^2 - \|Ax\|_{\mathcal{V}}^2$, we can then write

$$h(x) = \left(f^{\circ\circ}(x) + \frac{1}{2} \|x\|_{\mathcal{V}}^2 \right) - \frac{1}{2} \|x\|_{\mathcal{W}}^2. \quad (60)$$

The first part is l.s.c. by Theorem 2.3 and the latter is continuous, so h is always l.s.c.

To show that $g^{**} \leq h$, first note that

$$g^*(y) = \sup_x \langle x, y \rangle - f(x) - \frac{1}{2} \|Ax\|^2 \geq \sup_x \langle x, y \rangle - f(x) - \frac{1}{2} \|x\|^2 = f^\circ + \frac{1}{2} \|y\|^2.$$

Repeating this calculation gives $g^{**}(x) \leq f^{\circ\circ}(x) + \frac{1}{2} \|x\|^2 = h(x)$, as desired. The other part of (59) follows since $f^{\circ\circ} \leq f$, again by Theorem 2.3. The final statement is immediate by Proposition 2.2. \square

We now come to the main theorem of this section, inspired by Theorems 4.5 and 4.8 in [25]. We say that x is a local minimizer of g if there exists a neighborhood U of x in \mathcal{V} such that $g(y) \geq g(x)$ for all $y \in U$ and we say that x is a strict local minimizer of g if the inequality is strict for $y \neq x$.

Theorem 4.6. *Suppose that $\|A\| < 1$ and that f is a weakly l.s.c $[0, \infty]$ -valued functional. Let g be given by (57) and let h be given by (58). If x is a local minimizer (resp. strict local minimizer) of h , then it is also a local minimizer (resp. strict local minimizer) of g , and $h(x) = g(x)$. In particular the global minimizers coincide.*

Proof. Let x be a local minimizer of h . If

$$f^{\circ\circ}(x) = f(x) \quad (61)$$

does not hold, then Theorem 2.3 and Theorem 2.22 implies that there exists a unit vector ν such that

$$t \mapsto \left(f^{\circ\circ}(x + t\nu) + \frac{1}{2} \|x + t\nu\|_{\mathcal{V}}^2 \right)$$

is affine near $t = 0$. As in (60) it follows that

$$\begin{aligned} h(x + t\nu) &= f^{\circ\circ}(x + t\nu) + \frac{1}{2} \|A(x + t\nu) - d\|_{\mathcal{V}}^2 = \\ &\left(f^{\circ\circ}(x + t\nu) + \frac{1}{2} \|x + t\nu\|_{\mathcal{V}}^2 \right) - \frac{1}{2} \|x + t\nu\|_{\mathcal{W}}^2 - \operatorname{Re} \langle A(x + t\nu), d \rangle_{\mathcal{V}} + \frac{1}{2} \|d\|_{\mathcal{V}}^2, \end{aligned} \quad (62)$$

whose second derivative equals $-\|\nu\|_{\mathcal{W}}^2 < 0$ at $t = 0$, contradicting the assumption that x is a minimum of h (the inequality is strict since $\|A\| < 1$). We thus conclude that (61) holds, i.e. that $h(x) = g(x)$. In view of Proposition 4.5, it follows that x is a local minimizer also for g . The same argument applies to strict local minimizers.

We now prove that the global minimizers coincide. Note that global minimizers of g are global minimizers of h in view of (59) and the fact that $g(x) = g^{**}(x)$ for all global minimizers x . From this we also see that the global minimum of g and h coincide, let us denote this value by c . Reversely suppose that x is a global minimizer of h , i.e. $h(x) = c$. Then it is a local minimizer of g by the first part, which automatically is global for g since we otherwise would have $g(y) < c$ for some other value y . The proof is complete. \square

The situation when we only have $\|A\| \leq 1$ is a bit more involved, so we content with the following statement concerning the global minimizers.

Theorem 4.7. *Suppose that $\|A\| \leq 1$ and let f, g, h be as in the above theorem. Let G be the global minimizers of g and H the global minimizers of h . Then $G \subset H$, and each connected component of H contain points of G .*

Proof. The statement $G \subset H$ follows as in the above proof, as well as the fact that the global minimum of g and h coincide; we denote it by c .

If $x \in H$ and $g(x) > c$, then it follows by (62) that there exists a unit vector ν such that $\frac{d^2}{dt^2} h(x + t\nu) \leq 0$ in a neighborhood of $t = 0$. Strict inequality contradicts the assumption of global minima, so we deduce that $h(x + t\nu)$ is constant near $t = 0$, and hence (62) yields $\|\nu\|_{\mathcal{W}} = 0$, i.e. that ν lies in the kernel of the semi-norm $\|\cdot\|_{\mathcal{W}}$ (which is a linear subspace by convexity of the semi-norm). Let P be the affine hyperplane $P = x + \ker \|\cdot\|_{\mathcal{W}}$ and set $S = P \cap H$. For $y \in \ker \|\cdot\|_{\mathcal{W}}$, (62) implies that

$$h(x + y) = \left(S_{\mathcal{V}}(S_{\mathcal{V}}(f))(x + y) + \frac{1}{2} \|x + y\|_{\mathcal{V}}^2 \right) - \|x\|_{\mathcal{W}}^2 - \operatorname{Re} \langle A(x + t\nu), d \rangle_{\mathcal{V}} + \frac{1}{2} \|d\|_{\mathcal{V}}^2, \quad (63)$$

so Theorem 2.3 implies that h is convex on P . In particular, S is convex. Since h is l.s.c. it is also closed. Moreover S is bounded due to the quadratic term $\|x + y\|_{\mathcal{V}}^2$ in (63). S is therefore weakly closed, and hence it equals the closed convex hull of its extremal points, by the Krein Milman theorem. If x now is one of these extremal points, then we can argue as in the beginning of this proof and conclude that $h(x) = g(x)$, since the existence of a ν with the properties stated initially would contradict that x is an extremal point of S . \square

We end this section with some theoretical results concerning the l.s.c. convex envelope g^{**} of g , disregarding the computability aspect. We define \mathcal{V}_A to be \mathcal{V} equipped with the semi-norm $\|x\|_{\mathcal{V}_A} = \|Ax\|$. This makes \mathcal{V}_A into a semi-normed space (see e.g. Ch I.1 in [12]). The definition of the \mathcal{S} transform (11) is readily extended to this situation,

$$\mathcal{S}_{\mathcal{V}_A}(f)(y) := \sup_x -f(x) - \frac{1}{2} \|A(x - y)\|_{\mathcal{V}}^2, \quad (64)$$

and many of the results of part I can be generalized to include this case. In particular we have

Proposition 4.8. *Let \mathcal{V} be a finite dimensional space and let g be given by (57). Then*

$$g^{**} = \mathcal{S}_{\mathcal{V}_A}(\mathcal{S}_{\mathcal{V}_A}(f)) + \frac{1}{2} \|Ax - d\|_{\mathcal{V}}^2.$$

Proof. For clarity we denote the l.s.c. convex envelope of g by $CE(g)$ instead of g^{**} in this proof. $\|\cdot\|_{\mathcal{V}_A}$ is a norm on the vector space $(\ker A)^\perp$, which becomes a Hilbert space since it clearly is generated by a scalar product and \mathcal{V} is finite dimensional. We denote this Hilbert space by \mathcal{W} . By the equivalence of all norms in finite dimensional spaces, we have that a function on \mathcal{W} is l.s.c with respect to $\|\cdot\|_{\mathcal{V}_A}$ if and only if it is l.s.c. with respect to $\|\cdot\|_{\mathcal{V}}$. Set

$$\tilde{f}(x) = \inf_{y \in \ker A} f(x + y), \quad x \in \mathcal{W}.$$

Since \mathcal{W} is a Hilbert space we can apply the results from part I there. It is easy to see that both sides of the sought identity are constant on $\ker A$. More precisely, for $x \in \mathcal{W}$ and $y \in \mathcal{W}^\perp$ we have

$$CE(g)(x + y) = CE\left(\tilde{f}(\cdot) + \frac{1}{2} \|A \cdot - d\|^2\right)(x)$$

and $\mathcal{S}_{\mathcal{V}_A}(\mathcal{S}_{\mathcal{V}_A}(f))(x + y) = \mathcal{S}_{\mathcal{W}}(\mathcal{S}_{\mathcal{W}}(\tilde{f}))(x)$, so it suffices to prove

$$CE\left(\tilde{f}(\cdot) + \frac{1}{2} \|A \cdot - d\|^2\right)(x) = \mathcal{S}_{\mathcal{W}}(\mathcal{S}_{\mathcal{W}}(\tilde{f}))(x) + \frac{1}{2} \|Ax - d\|_2^2, \quad x \in \mathcal{W}. \quad (65)$$

Since

$$\frac{1}{2} \|Ax - d\|_2^2 = \frac{1}{2} \|x\|_{\mathcal{W}}^2 - \langle Ax, d \rangle_2 + \frac{1}{2} \|d\|_2^2$$

and the latter two terms are affine linear, they can be moved outside the parenthesis of CE in (65), whereby they cancel the corresponding terms from the right hand side. It follows that it suffices to prove (65) with $d = 0$. But this is immediate by Theorem 2.3. \square

4.4. ℓ_0 -minimization

This section is devoted entirely to the problem of minimizing

$$g(x) = \tau^2 \|x\|_0 + \frac{1}{2} \|Ax - d\|_2^2 \quad (66)$$

where $A \in \mathbb{M}_{m,n}$ and we assume that $\|A\| \leq 1$, which always is possible upon a rescaling (i.e. setting $\tau := c\tau$, $A := cA$ and $d := cd$ with $c = 1/\|A\|$). In practice it is common that $n \gg m$, so A will not be assumed injective. In this situation, it is not desirable to work with g^{**} since it is necessarily constant on $\ker A$, as noted before Proposition 4.8. Our main focus will thus

be on investigating methods similar to those of Section 4.3 to replace the discontinuous (66) with continuous functionals that have fewer local minima and the same global minima.

To begin with, however, we first derive an expression for $\mathcal{S}_{\mathcal{V}_A}(\tau^2 \|\cdot\|_0)(y)$. We adopt the same notation as around (64) and set $\mathcal{V} = \mathbb{R}^n$ equipped with the canonical scalar product. Let \mathcal{P}^n denote the collection of all subsets of $\{1, \dots, n\}$. Given $S \in \mathcal{P}^n$, let $A_S \in \mathbb{M}_{m,n}$ denote the matrix whose columns are those of A for column-indices in S and zeroes elsewhere. The method of least squares then states that $\inf_{x: \text{supp} x \in S} \|x - y\|_{\mathcal{V}_A}^2$ is solved by $\hat{x} = (A_S^* A_S)^\dagger A_S^* A y$, where $\text{supp} x$ denotes the support of x and $(A_S^* A_S)^\dagger$ denotes the Moore-Penrose inverse. Since $\text{supp} \hat{x} \in S$, by construction, we moreover have $A \hat{x} = A_S \hat{x} = A_S (A_S^* A_S)^\dagger A_S^* A y$ and it is easy to see that $A_S (A_S^* A_S)^\dagger A_S^*$ is the orthogonal projection (with canonical scalar product) onto the range of A_S , which we denote by $P_{\text{Ran} A_S}$. Summing up, we have

$$\sup_{\text{supp} x \in S} -\tau^2 \|x\|_0 - \frac{1}{2} \|x - y\|_{\mathcal{V}_A}^2 = -\tau^2 \#\text{supp} \hat{x} - \frac{1}{2} \|(P_{\text{Ran} A_S} - I) A y\|_2^2$$

where $\#\text{supp} \hat{x} = \#S$ unless \hat{x} has “additional zeroes”, in which case we get identity for the smaller subset \tilde{S} defined by $\tilde{S} = \#\text{supp} \hat{x}$. We conclude that

$$\begin{aligned} \mathcal{S}_{\mathcal{V}_A}(\tau^2 \|\cdot\|_0)(y) &= \sup_{S \in \mathcal{P}^n} \sup_{x: \text{supp} x \in S} -\tau^2 \|x\|_0 - \frac{1}{2} \|x - y\|_2^2 = \\ &= \sup_{S \in \mathcal{P}^n} -\tau^2 \#S - \frac{1}{2} \|P_{(\text{Ran} A_S)^\perp} A y\|_2^2. \end{aligned} \quad (67)$$

Although (67) is a beautiful closed form expression, to compute it is a combinatorial problem which is as hard minimizing the original functional (43). If the columns of A are assumed to be linearly independent, it is clear that it makes no sense to look for solutions with $\#S \geq m$, for then $(\text{Ran} A_S)^\perp = \emptyset$. The amount of candidates S for which one needs to compute (67) is then $\sum_{j=1}^{m-1} \binom{n}{j}$, which for $m < n/2$ can be estimated from above by the term $\binom{n}{m-1} \frac{n-m+2}{n-2m+3}$, clearly too many choices for most practical applications (we omit the details of this calculation).

On the other hand, if A is orthogonal, i.e. of the form $U\Sigma$ and $U^*U = I$, then $(\|\cdot\|_0)^\circ$ is readily computed. This follows by observing that $\|Ax\| = \|Dx\|$ so that

$$\tau^2 \|x\|_0 + \frac{1}{2} \|Ax\|^2 = \sum_{j=1}^n \tau^2 \chi_{\mathbb{R}}(x) + \frac{1}{2} \sigma_j^2 |x_j|, \quad (68)$$

and so Example 2.4 and Proposition 2.5 together imply that

$$\mathcal{S}_{\mathcal{V}_A}(\mathcal{S}_{\mathcal{V}_A}(\tau^2 \|\cdot\|_0))(x) = \sum_{j=1}^n \tau^2 - \left(\max\left\{ \tau - \frac{\sigma_j x_j}{\sqrt{2}}, 0 \right\} \right)^2. \quad (69)$$

We remark that the formula is valid also if some σ_j 's are 0. Equation (69) was first noted in [25]. In their notation (cf equation (4.1-4.2) of [25]), we have

$$S_{\mathcal{V}_A}(\mathcal{S}_{\mathcal{V}_A}(\tau^2 \|\cdot\|_0)) = \Phi_{CEL0}$$

where the latter is determined by parameters $\tau^2 = \lambda$ and $\|a_i\| = \sigma_i$. In particular, the symbol a_i denotes the i :th column of A , which clearly gives $\sigma_i = \|a_i\|$ for orthogonal A .

To deal with the situation of a general A , possibly having a large kernel, Section 4.3 suggests us to introduce another scalar product on $\mathcal{V} = \mathbb{C}^n$ by

$$\langle x, y \rangle_{\mathcal{V}} = \nu^2 \langle x, y \rangle_2,$$

where $\nu \geq \|A\|$. In this case

$$\mathcal{S}_{\mathcal{V}}(\mathcal{S}_{\mathcal{V}}(\tau^2 \|\cdot\|_0))(x) = \sum_{j=1}^n \tau^2 - \left(\max\left\{\tau - \frac{\nu x_j}{\sqrt{2}}, 0\right\} \right)^2 \quad (70)$$

and Theorem 4.6 applies, (or Theorem 4.7 in case $\nu = \|A\|$). The freedom of choosing ν leads us to the following dichotomy; On one hand, $\mathcal{S}_{\mathcal{V}_A}(\mathcal{S}_{\mathcal{V}_A}(\tau^2 \|\cdot\|_0))$ is not computable and even if it were, it would not be desirable to use it as pointed out in the beginning of this section (for A with large kernels at least). On the other hand, in the limit as $\nu \rightarrow \infty$ the expression (70) approaches

$$\tau^2 \|x\|_0 = \sum_{j=1}^n \tau^2 \chi_{\mathbb{R}}(x), \quad (71)$$

so the modified problem approaches the original one. It follows that it is desirable to keep ν as low as possible.

In [25] they have taken this one step further, suggesting to replace (71) with

$$\|x\|_{CE\ell_0} = \sum_{j=1}^n \tau^2 - \left(\max\left\{\tau - \frac{\|a_j\|_2 x_j}{\sqrt{2}}, 0\right\} \right)^2 \quad (72)$$

where a_j denotes the j :th column of A . This expression equals the $\mathcal{S}_{\mathcal{V}}\mathcal{S}_{\mathcal{V}}$ -transform if we introduce the scalar product

$$\langle x, y \rangle_{\mathcal{V}} = \sum_{j=1}^n \|a_j\|^2 x_j y_j.$$

However, for this choice of norm we do not in general have that A is a contraction, (as the simple example $A = [1 \ 1]$ shows), so the theorems from Section 4.3 do not apply. Theorem 4.5 in [25] however is very close to Theorem 4.7 of this paper, and Theorem 4.8 in [25] contains a similar statement regarding the local minima.

We now show that, upon slight modification of (72), the proofs of Theorem 4.6 and 4.7 can be modified to yield stronger conclusions. Let $\nu_j > \|a_j\|$ be any numbers and set

$$h(x) = \left(\sum_{j=1}^n \tau^2 - \left(\max\left\{\tau - \frac{\nu_j x_j}{\sqrt{2}}, 0\right\} \right)^2 \right) + \frac{1}{2} \|Ax - b\|_2^2 \quad (73)$$

Theorem 4.9. *Let g , h be given by (66) and (73). If x is a local minimizer (resp. strict local minimizer) of h , then it is also a local minimizer (resp. strict local minimizer) of g , and $h(x) = g(x)$. Moreover*

$$x_j \notin \left(0, \frac{\sqrt{2}\tau}{\nu_j}\right). \quad (74)$$

Finally, the set of global minimizers are nonempty and coincide.

Proof. Both h and g are l.s.c. and coercive, hence they have global minimizers. Clearly $h \leq g$. Let x be a local minimum for h . It is clear that $h(x) = g(x)$ if and only if (74) is fulfilled, and in this case x must be a minimum of g in view of $h \leq g$. So suppose (74) fails, and let j_0 be the index for which $0 < x_{j_0} < \frac{\sqrt{2}\tau}{\nu_j}$. From (73) we immediately deduce

$$\partial_{j_0}^2 h(x) = \|a_{j_0}\|^2 - \nu_{j_0}^2 < 0,$$

which contradicts that x is a local minimum of h . Moreover, if x in addition is a global minimum for h , then it is a global minimum for g , which follows immediately by the inequality $h \leq g$.

It remains to prove that if x is a global minimizer of g , it is also a global minimizer of h . Suppose not. This gives $h(y) < g(x)$ for some nearby point y , in view of the inequality $h \leq g$. It follows that there exists a global minimizer z for h with $h(z) < g(x)$. But then, by the first part of the proof $g(z) = h(z) < g(x)$, contradicting the choice of x . □

5. Appendix I; Milman's theorem for convex hulls

We prove Theorem 2.22, repeated below for convenience.

Theorem 2.22. *Let g be a weakly l.s.c. functional on a separable Hilbert space \mathcal{V} such that g^{**} is coercive. Given any $x \in \mathcal{V}$, we either have $g(x) = g^{**}(x)$ or there exists a unit vector ν and $t_0 > 0$ such that the function $h(t) = g^{**}(x_0 + t\nu)$ is affine on $(-t_0, t_0)$.*

To prove this we recall some concepts from [8]. A point x is called extremal if $x \in \text{dom } g^{**}$ and g^{**} is not affine on any relatively open segment containing x . Moreover g_{ext}^{**} denotes the functional which equals $g^{**}(x)$ for all extremal points x and ∞ else. The epigraph of any functional g is denoted by $[g]$. In the setting of [8] we let E be the separable Hilbert space \mathcal{V} with the weak topology. For an weakly l.s.c. functional g we then immediately have $g = g_{cl}$ in the terminology of [8] (the latter is the greatest l.s.c. minorant). Recall that g^{**} equals the l.s.c. convex envelope of g , see e.g. Proposition 13.39 of [5]. Since convex functionals are l.s.c. with respect to the weak topology if and only if they are with respect to the norm topology, (see Theorem 9.1 [5]), it follows that the l.s.c. convex envelope equals the weakly l.s.c. convex envelope, and in particular g^{**} equals f in Theorem 1 of [8]. Also, the level sets of g^{**} are closed and convex and if g^{**} is coercive they are also bounded. It follows that such level sets are compact in the weak topology, and hence, referring to the terminology of [8], that “ g is inf-compact in some direction” (with respect to the weak topology), namely the direction given by $\xi = 0$. With the above facts in mind, Theorem 1 of [8] takes the following form;

Theorem 5.1. *Let g be a weakly l.s.c. functional on a separable Hilbert space \mathcal{V} such that g^{**} is coercive, then*

$$[g_{ext}^{**}] \subset [g].$$

Based on this, we can now easily prove Theorem 2.22.

Proof of Theorem 2.22. Since $g \geq g^{**}$, Theorem 5.1 clearly implies that $g(x) = g^{**}(x)$ for all extremal points x for g^{**} . Consequently, if $g(x) = g^{**}(x)$ does not hold, then x is not extremal for g^{**} and the existence of ν follows by the definition of an extremal point for g^{**} . □

6. Appendix II; von-Neumann's trace inequality for operators

We repeat the statement of von-Neumann's trace inequality for operators on separable Hilbert spaces, which reads as follows.

Theorem 2.6. *Let $\mathcal{V}_1, \mathcal{V}_2$ be any separable Hilbert spaces, let $X, Y \in \mathcal{B}_2(\mathcal{V}_1, \mathcal{V}_2)$ be arbitrary and denote their singular values by $\sigma_j(X), \sigma_j(Y)$, respectively. Then*

$$\langle X, Y \rangle \leq \sum_j \sigma_j(X) \sigma_j(Y) \quad (75)$$

with equality if and only if the singular vectors can be chosen identically.

Surprisingly, this statement is nowhere to be found in the standard references, such as the books by Simon [23], Conway [11], Ringrose [21], Schatten [22], Dowson [14] Dunford and Schwartz [15]. The weaker inequality $\langle X, Y \rangle \leq \|X\| \|Y\|$, which is needed to show that the Hilbert-Schmidt class is a Hilbert space, is of course found in many of the above references, see e.g. Theorem 18, Section XI 6 of [15]. In fact, the “only if” part of the statement is hard to find even in the finite dimensional case. It does appear in [13] (cited 15 times at the time of writing) along with a discussion claiming that even von-Neumann himself had this part of the statement clear. We take the finite dimensional part of the statement for granted and proceed to prove the infinite dimensional case.

Proof. The case when one of the spaces is finite dimensional is easily reduced to the finite dimensional case, so we suppose that both are infinite dimensional. Since all separable Hilbert spaces are unitarily equivalent, we may assume that $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{V}$. Given any $Z \in \mathcal{B}_2$, let $(u_{Z,j})_{j=1}^\infty$ and $(v_{Z,j})_{j=1}^\infty$ be singular vectors, i.e. such that

$$Z = \sum_j \sigma_j(Z) u_{Z,j} \otimes v_{Z,j}$$

where $\sigma_j(Z)$ are the singular values and $u_{Z,j} \otimes v_{Z,j}(x) = u_{Z,j} \langle x, v_{Z,j} \rangle$ (see e.g. Theorem 1.4 [23]). Set $Z^J = \sum_{j=0}^J \sigma_j(Z) u_{Z,j} \otimes v_{Z,j}$. By the finite dimensional version of the theorem, we have

$$\langle X, Y \rangle = \lim_{J \rightarrow \infty} \langle X^J, Y^J \rangle \leq \lim_{J \rightarrow \infty} \sum_{j=0}^J \sigma_j(X^J) \sigma_j(Y^J) = \sum_{j=0}^\infty \sigma_j(X) \sigma_j(Y), \quad (76)$$

where the last inequality follows by the dominated convergence theorem. To see that the finite dimensional version applies, note that we may consider X^J and Y^J to be operators on the finite dimensional space \mathcal{W} spanned by all their singular vectors combined. In this space X^J and Y^J can be represented by matrices (upon choosing some orthonormal basis) and $\sigma_j(X^J) = 0$ for all j with $J < j \leq \dim \mathcal{W}$.

For the final part, note that it is immediate that equality in (75) holds if X and Y share singular vectors. Suppose now that this is not the case, but that equality in (75) holds anyway. For simplicity of notation set $\xi_j = \sigma_j(X)$. Let $J \leq K$ be such that

$$\xi_{J-1} < \xi_J = \xi_K < \xi_{K+1}$$

and define, for $\xi_{K+1} \leq s \leq \xi_{J-1}$,

$$X(s) = \sum_{j < J} \xi_j u_{X,j} \otimes v_{X,j} + \sum_{J \leq j \leq K} s u_{X,j} \otimes v_{X,j} + \sum_{j > K} \xi_j u_{X,j} \otimes v_{X,j}. \quad (77)$$

Holding Y fixed, it is clear that $\langle X(s), Y \rangle$ and $\sum_j \sigma_j(X(s)) \sigma_j(Y)$ are affine functions of s in the actual interval. Since the former is dominated by the latter and they equal at the interior point $s = \xi_J$, it follows that they must be the same (in $\xi_{K+1} \leq s \leq \xi_{J-1}$). In particular we have

$$\langle X(s), Y \rangle = \sum_j \sigma_j(X(s)) \sigma_j(Y) \quad (78)$$

for $s = \xi_{K+1}$. Consider now the (piecewise affine) extension of (77) defined by

$$X(s) = \sum_{j < J} \xi_j u_{X,j} \otimes v_{X,j} + \sum_{J \leq j} \min(s, \xi_j) u_{X,j} \otimes v_{X,j}$$

on $0 \leq s \leq \xi_{J-1}$. The earlier argument can now be bootstrapped to conclude that (78) holds for all $0 < s \leq \xi_{J-1}$. Upon taking a limit we conclude that (78) holds for $s = 0$, in which case $X(0)$ has finite rank. Repeating the entire argument with Y as the “variable” and $X(0)$ as the fixed matrix, we conclude that identity holds in (75) for $X^J = \sum_{j=0}^J \xi_j u_{X,j} \otimes v_{X,j}$ and $Y^{J'} = \sum_{j=0}^{J'} \eta_j u_{Y,j} \otimes v_{Y,j}$ where $\eta_j = \sigma_j(Y)$ and J' is any index such that $\eta_{J'-1} > \eta_{J'}$. By the finite dimensional version of the theorem, there are common singular vectors \tilde{u}_j and \tilde{v}_j such that $X^J = \sum_{j=0}^J \xi_j \tilde{u}_j \otimes \tilde{v}_j$ and $Y^{J'} = \sum_{j=0}^{J'} \eta_j \tilde{u}_j \otimes \tilde{v}_j$. It is easy to see that this contradicts the initial assumption that X and Y do not share singular vectors, and the proof is complete. \square

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